# CANADIAN JOURNAL OF MATHEMATICS

## *Journal Canadien de Mathématiques*

VOL. XII - NO. 2
1960

UNIVERSITY OF MICHIGAN
APR 25 1960
MATHEMATICS LIBRARY



*Published for*

THE CANADIAN MATHEMATICAL CONGRESS

*by the*

*University of Toronto Press*

EDITORIAL BOARD

H. S. M. Coxeter, G. F. D. Duff, R. D. James, R. L. Jeffery,
J.-M. Maranda, G. de B. Robinson, P. Scherk

*with the co-operation of*

D. B. DeLury, J. Dixmier, W. Fenchel, H. Freudenthal, I. Kaplansky,
N. S. Mendelsohn, C. A. Rogers, H. Schwerdtfeger, A. W. Tucker,
W. J. Webber, M. Wyman

The chief languages of the *Journal* are English and French.

Manuscripts for publication in the *Journal* should be sent to the *Editor-in-Chief*, G. F. D. Duff, University of Toronto. Authors are asked to write with a sense of perspective and as clearly as possible, especially in the introduction. Regarding typographical conventions, attention is drawn to the Author's Manual of which a copy will be furnished on request.

All other correspondence should be addressed to the *Managing Editor*, G. de B. Robinson, University of Toronto.

The *Journal* is published quarterly. Subscriptions should be sent to the *Managing Editor*. The price per volume of four numbers is $10.00. This is reduced to $5.00 for individual members of recognized Mathematical Societies.

The Canadian Mathematical Congress gratefully acknowledges the assistance of the following towards the cost of publishing this Journal:

| | |
|---|---|
| University of Alberta | Assumption University |
| University of British Columbia | Carleton College |
| Dalhousie University | Ecole Polytechnique |
| Université Laval | Loyola College |
| University of Manitoba | McGill University |
| McMaster University | Université de Montréal |
| Mount Allison University | Nova Scotia Technical College |
| Queen's University | St. Mary's University |
| University of Saskatchewan | University of Toronto |

National Research Council of Canada
and the
American Mathematical Society

AUTHORIZED AS SECOND CLASS MAIL, POST OFFICE DEPARTMENT, OTTAWA

# ON THE COMPOSITION OF BALANCED INCOMPLETE BLOCK DESIGNS

R. C. BOSE AND S. S. SHRIKHANDE

**Introduction.** The object of this paper is to develop a method of constructing balanced incomplete block designs. It consists in utilizing the existence of two balanced incomplete block designs to obtain another such design by what may be called the method of composition.

**1. Preliminary results on orthogonal arrays and balanced incomplete block designs.** Consider a matrix $A = (a_{ij})$ of k rows and $N$ columns, where each $a_{ij}$ represents one of the integers $1, 2, \ldots, s$. Consider all $t$-rowed submatrices of $N$ columns, which can be formed from this array, $t < k$. Each column of any $t$-rowed submatrix can be regarded as an ordered $t$-plet. The matrix will be called an orthogonal array $[N, k, s, t]$ of size $N$, $k$ constraints, $s$ levels, strength $t$, and index $\lambda$ if each of the $\binom{k}{t}$ $t$-rowed submatrices that can be formed from the array contains every one of the $s^t$ possible ordered $t$-plets exactly $\lambda$ times. Obviously $N = \lambda s^t$ and each row contains the integers $1, 2, \ldots, s$ exactly$\lambda\ s^{t-1}$ times. The idea of an orthogonal array is originally due to Rao **(16)** who utilized it in the construction of factorial arrangements in the design of experiments.

Denote by $f(\lambda s^t)$ the maximum number of constraints which are possible in an orthogonal array of size $\lambda s^t$, $s$ levels, strength $t$, and index $\lambda$. Then from Plackett and Burman **(15)** we have

THEOREM A. *For any $s$ and $\lambda$,*

$$f(\lambda s^2) < \left[\frac{\lambda s^2 - 1}{s - 1}\right]$$

*where $[x]$ is the largest integer not exceeding $x$.*

This inequality has been improved by Bose and Bush in **(2)**, where they also give methods of constructing orthogonal arrays of strength two and three when the number of levels $s$ is a prime power.

Let a set of $s$ distinct symbols be arranged in an $s \times s$ square in such a way that every symbol occurs exactly once in every row and once in every column. Such a square is called a Latin square of order $s$. Two Latin squares of order

---

Received August 30, 1959. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. AF 49(638)-213. Reproduction in whole or in part is permitted for any purpose of the United States Government.

$s$ are called orthogonal, if when one of the squares is superposed on the other, every symbol of the first square occurs with every symbol of the second square once and only once. A set of Latin squares of order $s$ is said to be a set of mutually orthogonal Latin squares (m.o.l.s.) if any two of them are orthogonal. It is known **(2)** that the existence of $k$-2 m.o.l.s. of order $s$ is equivalent to the existence of an orthogonal array $[s^2, k, s, 2]$. Hence for any $s$, $f(s^2) < s + 1$ implies that $N(s) < s - 1$, where by $N(s)$ we denote the maximum number of m.o.l.s. of order $s$. If $s$ is a prime power, then it is known **(10; 11)** that $N(s) = s - 1$. If $v = p_1^{n_1} p_2^{n_2} \dots p_u^{n_u}$ is the prime power decomposition of an integer $v$, and we define $n(v) = \min(p_1^{n_1}, p_2^{n_2}, \dots, p_u^{n_u}) - 1$, then MacNeish **(10)** and Mann **(11)** showed that there exists a set of at least $n(v)$ m.o.l.s. of order $v$, that is, $N(v) \geqslant n(v)$. Recently Parker **(13)** showed how in certain cases one could construct more than $n(v)$ m.o.l.s. of order $v$. Parker's method has been generalized by the present authors who showed **(3; 4)** in particular that Euler's conjecture about the non-existence of two orthogonal Latin squares of order 2(mod 4) is false for an infinity of values of $v \geqslant 22$. By using the method of differences Parker **(14)** later on showed that $N(v) \geqslant 2$ for $v = \frac{1}{2}(3q - 1)$, where $q$ is a prime power $\equiv 3 \pmod 4$. In a joint paper with Parker **(5)** the present authors have shown that Euler's conjecture is false for all values of $v \geqslant 10$.

We call an array $[\lambda s^t, k, s, t]$ $\alpha$-resolvable if the $\lambda s^t$ columns can be separated into $\lambda s^{t-1}/\alpha$ sets of $\alpha s$ each, such that in each set every row contains each of the $s$ integers $1, 2, \dots, s$ exactly $\alpha$ times. A 1-resolvable array is called resolvable. Suppose there exists a set $\sum$ of $k - 1$ m.o.l.s. of order $s$, then we can take $\sum$ in the standard form in which the first row of each Latin square contains the integers $1, 2, \dots, s$ in that order. We now prefix to the set $\sum$ a $s \times s$ square containing the integer $i$ in each position in the $i$th column. If we then write down the elements of each square in a single row such that the integer in the $i$th row and the $j$th column occupies the $n$th position, where $n = s(i - 1) + j$; $i, j = 1, 2, \dots, s$; then we get an orthogonal array $[s^2, k, s, 2]$ which is resolvable. We thus have the following theorem which is essentially contained in **(6)**.

THEOREM B. *Existence of* $k - 1$ m.o.l.s. *of order* $s$ *implies the existence of a resolvable array* $[s^2, k, s, 2]$.

A balanced incomplete block design (BIB) **(18)** with parameters $v, b, r, k, \lambda$ is an arrangement of $v$ objects or treatments in $b$ sets or blocks such that (i) every block contains $k < v$ different objects and (ii) every pair of treatments occurs in $\lambda$ blocks. Then it is easy to see that each treatment occurs in exactly $r$ blocks and the parameters satisfy the relations

$$\lambda(v - 1) = r(k - 1), \quad bk = vr, \quad b \geqslant v. \tag{1.1}$$

The last inequality is due to Fisher **(7)**. A BIB design is called symmetrical if $b = v$ and hence $k = r$. It will be called $\beta$-resolvable if the blocks can be

separated into sets such that each set contains every treatment exactly $\beta$ times. A 1-resolvable BIB design is called resolvable. A BIB design with parameters $v, k, \lambda$ will be denoted by BIB $(v; k; \lambda)$ and if $\lambda = 1$ by BIB $(v; k)$. If the design is resolvable we denote it by RBIB $(v; k; \lambda)$ and RBIB $(v; k)$ respectively.

A BIB design is called separable if its blocks can be divided into sets of type I or II **(4)**.

From Theorem 2 in **(4)** we have:

THEOREM C. *If there exists a* BIB $(v; k)$ *then*

$$N(v) \geqslant N(k) - 1.$$

*Further, if the design is separable then*

$$N(v) \geqslant N(k).$$

From Theorem 2 of **(5)** and corollary of Theorem 12 of **(4)** we have

THEOREM D. *Existence of* BIB $(v; k)$ *implies that*

$$N(v-1) \geqslant \min(N(k),\quad 1 + N(k-1)) - 1.$$

*Further, if the design is resolvable, then*

$$N(v-1) \geqslant \min(N(k)),\quad N(k-1)).$$

**2. Pairwise balanced designs of index** $\lambda$. An arrangement of $v$ treatments in $b$ blocks will be called a pairwise balanced design (D) of index $\lambda$, if each block contains either $k_1, k_2, \ldots,$ or $k_m$ treatments which are all distinct $(k_i \neq k_j \leqslant v)$, and every pair of treatments occurs in exactly $\lambda$ blocks. Such a design will be said to be of type $(v; k_1, \ldots, k_m; \lambda)$. If the number of blocks containing $k_i$ treatments is $b_i$, then

$$b = \sum_1^m b_i, \lambda v(v-1) = \sum_1^m b_i k_i (k_i - 1).$$

The subdesign $(D_i)$ formed by the blocks of size $k_i$, will be called the $i$th equiblock component of (D), $i = 1, 2, \ldots, m$.

A subset of blocks of $(D_i)$ will be said to be of general type I, if every treatment occurs in the subset $\alpha k_i$ times, where $\alpha$ is a divisor of $\lambda$. The number of blocks in the subset is clearly $\alpha v$. As proved in **(8; 17)**, we can arrange the treatments within the blocks of the subset in such a way that every treatment comes in each position exactly $\alpha$ times. If the blocks are written as columns, each treatment occurs $\alpha$ times in every row. When so written out the blocks will be said to be in the standard form.

A subset of blocks of $(D_i)$ will be said to be of general type II if every treatment occurs in the subset exactly $\beta$ times when $\beta$ is a divisor of $\lambda$. The set of blocks will be said to form a $\beta$-replicate. The number of blocks in such a subset is clearly $(\beta v)/(k_i)$.

The component $(D_i)$ is said to be separable in the general sense if the blocks of $(D_i)$ can be divided into subsets of general type I or II. (Both types may occur in $(D_i)$ at the same time.) If $\alpha = \beta = 1$, then $(D_i)$ is called separable **(4)**.

If each $(D_i)$ is separable in the general sense with $\alpha$ and $\beta$ independent of $i$, then $(D)$ is called separable in the general sense. If each $(D_i)$ is separable then $(D)$ will be called separable.

The set of equiblock components $(D_1), (D_2), \ldots, (D_l)$ will be said to form a clear set if $\sum_1^l b_i$ blocks comprising $(D_1), (D_2), \ldots, (D_l)$ are such that no two blocks of the set have a treatment in common. Clearly a necessary condition for this is

$$\sum_1^l b_i k_i \leqslant v.$$

## 3. Use of pairwise balanced design in the construction of orthogonal arrays.

THEOREM 1. *Let there exist a pairwise balanced design* $(D)$ *of type* $(v; k_1, \ldots, k_m; \lambda)$ *and suppose that there exist* $q_i - 1$ m.o.l.s. *of order* $k_i$, $i = 1, 2, \ldots, m$. *Put*

$$q = \min(q_1, q_2, \ldots, q_m).$$

*Then*

$$f(\lambda v^2) \geqslant q.$$

*If the set of equiblock components* $(D_1), (D_2), \ldots, (D_l)$ *form a clear set, and*

$$q^* = \min(q_1 + 1, q_2 + 1, \ldots, q_l + 1, q_{l+1}, \ldots, q_m)$$

*then* $f(\lambda v^2) \geqslant q^*$. *If the design* $(D)$ *is separable in the general sense, then*

$$f(\lambda v^2) \geqslant q + 1$$

*and we can construct* $A = [\lambda v^2, q, v, 2]$ *which is* $\lambda$*-resolvable. If in particular* $(D)$ *is separable then* $A$ *is resolvable.*

*Proof.* Proof follows the general lines of Theorem 1 in **(4; 5)**. Let the treatments of the design be $t_1, t_2, \ldots, t_v$ and let the blocks of the design (written out as columns) belonging to the equiblock component $(D_i)$ be

$$\delta_{i1}, \delta_{i2}, \ldots, \delta_{ib_i} \qquad (i = 1, 2, \ldots, m).$$

Define the $k_i \times b_i$ matrix $D_i$ by

$$D_i = [\delta_{i1}, \delta_{i2}, \ldots, \delta_{ib_i}].$$

Let $P_i$ be the matrix of order $q_i \times k_i(k_i - 1)$ defined in Lemma 2 of **(4)**, the elements of $P_i$ being the symbols $1, 2, \ldots, k_i$. Let $P_{ic}$, $c = 1, 2, \ldots, k_i - 1$ be the submatrices of $P_i$, such that each row of $P_{ic}$ contains the symbols

$1, 2, \ldots, k_i$, exactly once. Define $P_i(\delta_{iu})$ in the same manner as in Lemma 2 of **(4)** and let

$$P_i(D_i) = [P_i(\delta_{i1}), P_i(\delta_{i2}), \ldots, P_i(\delta_{ib_i})].$$

Then $P_i(D_i)$ is of order $q_i \times b_i k_i(k_i - 1)$. If $t_a$ and $t_b$ are any two treatments occurring in the same block $(\delta_{iu})$, then the ordered pair $t_a, t_b$ occurs exactly once as a column in any two-rowed submatrix of $P_i(\delta_{iu})$. Let $\Delta_i$ be the matrix obtained from $P_i(D_i)$ by retaining only the first $q$ rows, and let

$$\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_m).$$

Then from (2.1), $\Delta$ is of order $q \times \lambda v(v - 1)$ and since any two treatments occur exactly in $\lambda$ blocks of $(D)$, any two-rowed submatrix of $\Delta$ contains exactly $\lambda$ columns of ordered pairs of any two distinct treatments chosen from $t_1, t_2, \ldots, t_v$. Let $\Delta_0$ be a $q \times \lambda v$ matrix containing $t_i$ in all positions in columns numbered from $(i - 1)\lambda + 1$ to $i\lambda$, $i = 1, 2, \ldots, v$. Then the matrix $(\Delta_0, \Delta)$ obviously gives $A = [\lambda v^2, q, v, 2]$.

The second part of the theorem can be proved along the same lines as Theorem 1 in **(5)**.

To prove the last part of the theorem we note that each $(D_i)$ can be broken up into $x_i$ sets of $\alpha v$ blocks of general type I and $y_i$ sets of $\beta v/k_i$ blocks of general type II, where $\alpha$ and $\beta$ are divisors of $\lambda$ and are the same for each $(D_i)$. Thus each treatment occurs $\alpha k_i x_i + \beta y_i = r_i$ times in $(D_i)$, $i = 1, 2, \ldots, m$. Following the proof of the latter part of Theorem 1 in **(4)**, it is easily seen that the columns of $\Delta$ can be divided into $\sum_1^m x_i k_i(k_i - 1)$ sets of $\alpha v$ and $\sum^m y_i(k_i - 1)$ sets of $\beta v$ columns respectively, where in each set every row contains all the treatments exactly $\alpha$ and $\beta$ times respectively.

If $\alpha = \beta = 1$, then $(\Delta_0, \Delta)$ is a resolvable array $[\lambda v^2, q, v, 2]$. We can now add an additional row by putting $t_i$ in the $q + 1$th position under $\lambda$ sets of $v$ columns of $(\Delta_0, \Delta)$, $i = 1, 2, \ldots, v$. This gives an array $[\lambda v^2, q + 1, v, 2]$.

Now consider the case where both $\alpha$ and $\beta$ are not equal to 1. Since

$$\begin{aligned}\alpha v \sum_1^m x_i k_i(k_i - 1) + \beta v \sum_1^m y_i(k_i - 1) &= v \sum_1^m (k_i - 1)\,[\alpha k_i x_i + \beta y_i] \\ &= v \sum_1^m (k_i - 1) r_i \\ &= \sum_1^m (k_i - 1)\, k_i b_i \\ &= \lambda v(v - 1),\end{aligned}$$

$$\alpha \sum_1^m x_i k_i(k_i - 1) + \beta \sum_1^m y_i(k_i - 1) = \lambda(v - 1). \tag{3.1}$$

Let $\lambda = p\alpha = p'\beta$, say, and let

$$\sum_1^m x_i k_i(k_i - 1) = pc + d, \qquad c > 0, 0 < d < p \tag{3.2}$$

and

$$\sum_{1}^{m} y_i(k_i - 1) = p'c' + d', \qquad c' \geqslant 0, 0 \leqslant d' < p'. \tag{3.3}$$

Then (3.1) gives

$$\alpha[pc + d] + \beta[p'c' + d'] = \lambda(v - 1)$$

or

$$\lambda(c + c') + d\alpha + d'\beta = \lambda(v - 1). \tag{3.4}$$

Since $d\alpha < p\alpha = \lambda$, $d'\beta < p'\beta = \lambda$, (3.4) implies that

$$d\alpha + d'\beta = \lambda. \tag{3.5}$$

From (3.2), it is clear that $\sum x_i k_i(k_i - 1)$ sets of $\alpha v$ columns of $\Delta$ can be separated into $c$ sets of $\lambda v$ columns, each set containing every one of the $v$ treatments exactly $\lambda$ times in any row and another set containing $d\alpha v$ columns in which every treatment occurs exactly $d\alpha$ times in every row. Similarly from (3.4) we get $c'$ sets of $\lambda v$ columns containing each treatment $\lambda$ times in every row and a set of $d'\beta v$ columns contains each treatment exactly $d'\beta$ times in a row. Combining the sets of $d\alpha v$ and $d'\beta v$ columns we get $\lambda v$ columns containing each treatment $\lambda$ times in every row. Thus the columns of $\Delta$ are divisible into $(v - 1)$ sets each of $\lambda v$ columns, such that in each set every row contains all the $v$ treatments exactly $\lambda$ times. It is now obvious that $(\Delta_0, \Delta)$ is an array $[\lambda v^2, q, v, 2]$ which is $\lambda$-resolvable. We can now add an additional row by placing $t_i$ in the $(q + 1)$th position under the $i$th set $i = 1, 2, \ldots, v$, thus giving $[\lambda v^2, q + 1, v, 2]$.

COROLLARY 1.1. *Existence of* BIB $(v; k; \lambda)$ *and the existence of* $q - 1$ m.o.l.s. *of order* $k$ *implies that*

$$f(\lambda v^2) \geqslant q.$$

*Further, if the design is separable in the general sense, then*

$$f(\lambda v^2) \geqslant q + 1$$

*and we can construct* $A = [\lambda v^2, q, v, 2]$ *which is* $\lambda$*-resolvable. If the design is separable, then* $A$ *is resolvable.*

## 4. Composition of blanced incomplete block designs.

THEOREM 2A. *If* BIB $(v_1; k; \lambda_1)$ *and* BIB $(v_2; k; \lambda_2)$ *exist and if* $f(\lambda_2 v_2^2) \geqslant k$, *then* BIB $(v_1 v_2; k; \lambda_1 \lambda_2)$ *exists.*

*Proof.* Let the two designs be denoted by $(D_1)$ and $(D_2)$ respectively. Write the blocks of each design as columns. Then $(D_1)$ can be written as a matrix of $k$ rows and $b_1$ columns, where $b_1$ is the number of blocks in $(D_1)$. Let the treatments of $(D_1)$ be $t_1, t_2, \ldots, t_v$; and let $A$ be an orthogonal array

$A = [\lambda_2 v_2^2, k, v_2, 2]$ in the integers $1, 2, \ldots, v_2$. Let $(\beta_i)$ be any block of $(D_1)$ containing treatments

$$t_{i_1}, t_{i_2}, \ldots, t_{i_k}$$

in positions $1, 2, \ldots, k$ respectively. Whenever $j$ ($j = 1, 2, \ldots, v_2$) occurs in row $p$ of $A$, replace it by

$$t_{i_p, j}, \qquad p = 1, 2, \ldots, k.$$

We thus get a matrix $A(\beta_i)$ of $k$ rows and $\lambda_2 v_2^2$ columns. Define

$$A(D_1) = (A(\beta_1), \ldots, A(\beta_{b_1})).$$

Then $A(D_1)$ is a matrix of $k$ rows and $b_1\lambda_2 v_2^2$ columns in which the entries are the $v_1 v_2$ symbols $t_{i,j}$, $i = 1, 2, \ldots, v_1$; $j = 1, 2, \ldots, v_2$. If $c \neq c'$, the treatments $t_c$ and $t_{c'}$ occur together in exactly $\lambda_1$ blocks of $(D_1)$. Suppose $(\beta)$ is a block of $(D_1)$ containing $t_c$ and $t_{c'}$ in positions $i$ and $i'$ respectively. In $A$ integers $j, j'$ occur in positions $i$ and $i'$ respectively in exactly $\lambda_2$ columns. Hence treatments $t_{c,j}$ and $t_{c',j'}$ occur together in the corresponding $\lambda_2$ columns of $A(\beta)$. Obviously then these treatments will occur in $\lambda_1\lambda_2$ columns of $A(D_1)$. We note that this is true whether or not $j$ and $j'$ are equal. Thus the $v_1 v_2$, treatments $t_{i,j}$ can be divided into $v_1$ sets $(t_{i,1}, t_{i,2}, \ldots, t_{i,v_2})$, $i = 1, 2, \ldots, v_1$, such that any two treatments coming from different sets occur exactly $\lambda_1\lambda_2$ times in the blocks (columns) of $A(D_1)$. We now take $\lambda_1$ repetitions of the design $(D_2)$ for each of these $v_1$ sets of $v_2$ treatments. The totality of blocks thus obtained obviously provide BIB $(v_1 v_2; k; \lambda_1\lambda_2)$.

COROLLARY 2A.1. *If* BIB $(v_1; k)$ *and* BIB $(v_2; k)$ *exist and* $N(v_2) > k - 2$, *then* BIB $(v_1 v_2; k)$ *exists.*

Using the above corollary and Theorem C, we have the following result due to Skolem, given in the notes to Netto's book **(12)**.

COROLLARY 2A.2. *If $k$ is a prime power and* BIB $(v_1; k)$ *and* BIB $(v_2; k)$ *exist, then* BIB $(v_1 v_2; k)$ *exists.*

THEOREM 2B. *If separable designs* BIB $(v_1; k; \lambda_1)$ *and* BIB $(v_2; k; \lambda_2)$ *exist and if a resolvable array* $A = [\lambda_2 v_2^2, k, v_2, 2]$ *exists, then a separable design* BIB $(v_1 v_2; k; \lambda_1\lambda_2)$ *exists. If in particular the original designs are resolvable so is the obtained design.*

*Proof.* Suppose that the first design $(D_1)$ can be separated into sets $S_1, S_2, \ldots, S_x$ of type I and $S^*_1, S^*_2, \ldots, S^*_{x'}$ of type II. Then obviously $xk + x' = r_1$, the number of replications of any treatment in $(D_1)$. The sets $S_q$ each contain $v_1$ blocks and the sets $S^*_{q'}$ each contain $v_1/k$ blocks. Without loss of generality assume that each set $S_q$ is put in the standard form, that is, in each row of $S_q$ every treatment of $(D_1)$ occurs exactly once.

Since $A$ is resolvable, we can put

$$A = (A_1, A_2, \ldots, A_{\lambda_2 v_2})$$

where each row of $A_i$ contains all the integers $1, 2, \ldots, v_2$ exactly once. As in the previous theorem, let

$$A(D_1) = (\ldots, A_p(S_q), \ldots, A_p(S^*_{q'}), \ldots,)$$

where $p = 1, 2, \ldots, \lambda_2 v_2$; $q = 1, 2, \ldots, x$; $q' = 1, 2, \ldots, x'$. Then it is easily seen that with respect to the $v_1v_2$ treatments $t_{i,j}$ defined in the previous theorem, each set $A_p(S_q)$ gives a set of $v_1v_2$ blocks of type I and each set $A_p(S^*_{q'})$ gives a set of $(v_1v_2)/k$ blocks of type II. Taking the additional blocks obtained from $\lambda_1$ repetitions of the separable design $(D_2)$ for each of the $v_1$ sets of $v_2$ treatments as in the previous theorem, we get a separable design BIB $(v_1v_2; k; \lambda_1\lambda_2)$. It is obvious that if in particular the original designs are resolvable so is the new design for $v_1v_2$ treatments.

Taking $\lambda_1 = \lambda_2 = 1$ and the particular case of resolvability, we get from Theorem B,

Corollary 2B.1. *If* RBIB $(v_1; k)$ *and* RBIB $(v_2; k)$ *exist and* $N(v_2) > k-1$, *then* RBIB $(v_1v_2; k)$ *exists.*

Using Theorem C, the above gives

Corollary 2B.2. *If $k$ is a prime power and* RBIB $(v_1; k)$ and RBIB $(v_2; k)$ *exist, then* RBIB $(v_1v_2; k)$ *exists.*

Theorem 2C. *If* BIB $(v_1; k; \lambda_1)$ BIB $(v_2; k; \lambda_2)$ *exist and* $f(\lambda_2(v_2 - 1)^2) > k$, *then* BIB $(v_1(v_2 - 1) + 1; k; \lambda_1\lambda_2)$ *exists.*

*Proof.* Let $(D_1)$ be the design with $v_1$ treatments and let $A_1 = [\lambda_2(v_2 - 1)^2, k, v_2 - 1, 2]$. Then as in Theorem 2A, the matrix $A_1(D_1)$ gives blocks of size $k$ in which any two treatments coming from different sets

$$(t_{i,1}, \ldots, t_{i,v_2-1}), \qquad i = 1, 2, \ldots, v_1,$$

occur together in exactly $\lambda_1\lambda_2$ blocks. Take a new treatment say $\theta$, and consider the $v_1$ sets of $v_2$ treatments

$$(\theta, t_{i,1}, \ldots, t_{i,v_2-1}), \qquad i = 1, 2, \ldots, v_1.$$

The $\lambda_1$ repetitions of the design $(D_2)$ with each of the $v_1$ sets of $v_2$ treatments above together with the blocks $A_1(D_1)$ give the BIB $(v_1(v_2 - 1) + 1; k; \lambda_1\lambda_2)$.

Corollary 2C.1. *If* BIB $(v_1; k)$ *and* BIB $(v_2; k)$ *exist and* $N(v_2 - 1) > k-2$, *then* BIB $(v_1(v_2 - 1) + 1; k)$ *exists.*

The particular case of this corollary when $v_2 = k$ is given in the notes added by Skolem in **(12)**.

Using Theorem D and the above corollary we have

Corollary 2C.2. *If $k$ and $k - 1$ are both prime powers, then the existence of* BIB $(v_1; k)$ *and* BIB $(v_2; k)$ *implies the existence of* BIB $(v_1(v_2 - 1) + 1; k)$.

THEOREM 2D. *If* BIB $(v; k; \lambda)$ *exists and* $k - 1$ *is a prime power, then* BIB $((k-2)v + 1; k - 1; \lambda)$ *exists which is* $\lambda$*-resolvable.*

*Proof.* Let the remaining parameters of $(D)$ the BIB $(v; k; \lambda)$ be $b$ and $r$. Since $k - 1$ is a prime power, there exists an orthogonal array $A_1 = [(k-1)^2, k, k-1, 2]$ in $k-1$ integers $1, 2, \ldots, k-1$. Without loss of generality assume that the first column of $A_1$ consists entirely of 1's. Using Theorem 2C with $v_1 = v$, $\lambda_1 = \lambda, v_2 = k, \lambda_2 = 1$, we obtain a BIB design with parameters $v^* = v(k-1) + 1$, $b^* = b(k-1)^2 + \lambda v$, $r^* = \lambda v$, $k^* = k$, $\lambda^* = \lambda$. From this design omit the $b$ blocks of $A_1(D)$ which arise from the first column of $A_1$. Obviously these blocks form a BIB $(v; k; \lambda)$ for the treatments $t_{1,1}, \ldots, t_{v,1}$, where $t_1, t_2, \ldots, t_v$ are the treatments of $(D)$ In the design $(D_1)$ formed of the remaining $b(k-1)^2 + \lambda v - b$ blocks, each of the treatments $t_{1,1}, \ldots, t_{v,1}$ occurs $\lambda v - r$ times. Further, no two of these treatments can occur in the same block of $(D_1)$. Since from (1.1)

$$\begin{aligned} b(k-1)^2 + \lambda v - b &= bk^2 - 2bk + \lambda v \\ &= v(rk - 2r + \lambda) \\ &= v(\lambda v - r) \end{aligned}$$

the blocks of $(D_1)$ can be separated into $v$ sets of $(\lambda v - r)$ blocks, such that each block of the $i$th set contains the treatment $t_{i,1}$, $i = 1, 2, \ldots, v$. In this set $t_{i,1}$ obviously occurs $\lambda$ times with each of the remaining $v(k-2) + 1$ treatments excepting $t_{j,1}$, $j \neq i = 1, 2, \ldots, v$. Omitting the treatment $t_{i,1}$ from the blocks of the $i$th set $i = 1, 2, \ldots, v$, we get BIB $((k-2)v + 1; k-1; \lambda)$ which is obviously $\lambda$-resolvable.

COROLLARY 2D.1. *If* BIB $(v; k)$ *exists and* $k - 1$ *is a prime power then* RBIB $((k-2)v + 1; k - 1)$ *exists.*

We note that in actual applications of the above theorems the trivial existence of RBIB $(k; k)$ is very useful.

**5. BIB designs with** $k = 5$, $\lambda = 1$. A BIB design with $k = 5$, $\lambda = 1$ belongs to one of the two series **(1)**

$(G_1)$ $\quad v = 20t + 1, b = t(20t + 1), r = 5t$

$(G_2)$ $\quad v = 20t + 5, b = (5t + 1)(4t + 1), r = 5t + 1.$

If $v = 20t + 1$, we denote the corresponding design by $G_1(v)$. Similarly if $v = 20t + 5$, the corresponding design is denoted by $G_2(v)$. When $v$ is of the form $20t + 1$ or $20t + 5$ the corresponding design will be denoted by $G(v)$. Using the results in **(1)** and on p. 118 of **(11)** and the corollaries 2A.2, 2B.2, 2C.2, we can state the following theorem.

THEOREM 3. (a) *If* $v = 20t + 1$ *is a prime power and* $x$ *is a primitive element of* $GF(v)$ *and if* $x^{4t} + 1 = x^q$, $q$ *odd then* $G_1(v)$ *exists.*

(b) *If $v = 20t + 5$ and $4t + 1$ is a prime power, then $G_2(v)$ exists.*

(c) *Existence of $G(v)$ implies the existence of $G(5v)$, and if $G(v)$ is resolvable so is $G(5v)$.*

(d) *If $G(v_1)$ and $G(v_2)$ exist, then $G(v_1v_2)$ exists, and if $G(v_1)$ and $G(v_2)$ are both resolvable so is $G(v_1v_2)$.*

(e) *Existence of $G(v)$ implies the existence of $G(4v + 1)$ and $G(5v - 4)$.*

(f) *Existence of $G(v_1)$ and $G(v_2)$ implies the existence of $G(v_1(v_2 - 1) + 1)$.*

From (a) using the powers of primitive roots of primes given in **(9)** solutions for $v = 41, 61, 241, 281, 641, 701, 881$ can be obtained. From (b) we get solutions for $v = 25, 45, 65, 85, 125, 145, 185, 205, 245, 265, 305, 365, 405, 445, 485, 505, 545, 565, 605, 625, 685, 745, 785, 845, 865, 905, 965, 985$. Similarly (c), (d), (e), and (f) provide solutions for a large number of values of $v$. Using these methods, solutions for all $v$ ($< 1000$) of the form $20t + 1$ or $20t + 5$ can be obtained excepting for 81, 141, 161, 285, 345, 361, 381, 385, 461, 465, 541, 561, 585, 645, 665, 681, 705, 761, 765, 781, 801, 941, 961, 981. We note that resolvable solutions for $v = s^q$, $q > 2$ can always be constructed.

**6. BIB designs with $k = 4, \lambda = 1$.** BIB designs with $k = 4, \lambda = 1$ can be classified in two series:

$(F_1)$: $\quad v = 12t + 1, b = t(12t + 1), r = 4t$

$(F_2)$: $\quad v = 12t + 4, b = (4t + 1)(3t + 1), r = 4t + 1.$

If $v = 12t + 1$, we denote the corresponding solution by $F_1(v)$ and if $v = 12t + 4$ we denote it by $F_2(v)$. $F(v)$ will denote the design when $v$ is of the form $12t + 1$ or $12t + 4$. Using the results in **(1)** and on page 118 of **(11)** and the corollaries 2A.2, 2B.2, 2C.2, 2D.1, we have the following theorem.

THEOREM 4.

(a) *If $v = 12t + 1$ is a prime power and $x$ is a primitive element of $GF(v)$* and $x^{4t} - 1 = x^q$, *$q$ odd, then $F_1(v)$ exists.*

(b) *If $v = 12t + 4$ and $4t + 1$ is a prime power, then a resolvable solution for $F_2(v)$ exists.*

(c) *Existence of $F(v)$ implies the existence of $F(4v)$, and if $F(v)$ is resolvable so is $F(4v)$.*

(d) *If $F(v_1)$ and $F(v_2)$ exist, then $F(v_1v_2)$ exists, and if $F(v_1)$ and $F(v_2)$ are resolvable, so is $F(v_1v_2)$.*

(e) *Existence of $F(v)$ implies the existence of $F(3v + 1)$ and $F(4v - 3)$.*

(f) *Existence of $F(v_1)$ and $F(v_2)$ implies the existence of $F(v_1(v_2 - 1) + 1)$.*

(g) *Existence of $G(v)$ implies the existence of a resolvable solution for $F(3v+1)$.*

From (a) above with the help of **(9)** we get solutions for $v = 13, 25, 73, 181, 277, 409, 457, 541, 709$; from (b) resolvable solutions for $v = 16, 28, 40, 52, 76, 88, 112, 124, 148, 160, 184, 220, 244, 268, 292, 304, 328, 340, 364, 376,$

412, 448, 472, 508, 520, 544, 580, 592, 688, 700, 724, 772, 808, 832, 844, 868, 880, 940, 952 are obtained. From (*g*) we get resolvable solutions for $v = 136$, 196, 256, 316, 436, 556, 604, 616, 664, 676, 796, 904, 916, 964. Resolvable solutions for $v = 64$, 208, 352, 496, 640, 736, 784, 976 are provided by (c). Similarly (d), (e), and (f) give solutions for a large number of values of $v$. It has not been possible to obtain solutions for $v < 1000$ for the following values: 37, 133, 145, 172, 217, 232, 280, 361, 424, 460, 469, 505, 517, 529, 532, 565, 568, 577, 613, 649, 652, 685, 697, 712, 745, 748, 841, 853, 856, 865, 889, 892, 901, 925, 928, 997.

**7. Concluding remarks.** The BIB designs with $k = 5, \lambda = 1$ and RBIB designs with $k = 4, \lambda = 1$ are especially interesting from the point of view of constructing orthogonal Latin squares. From Theorem 3 **(5)** it follows that existence of $G(v)$ with $v = 20t + 1$ and $20t + 5$ implies the existence of at least two orthogonal Latin squares of order $20t - 2$ and $20t + 2$, respectively, which are of the form 2(mod 4). Similarly from Theorem 4 of **(5)** existence of resolvable solution $F_2(12t + 4)$, $t > 5$, coupled with the fact that $N(v) > 2$, for $v = 10$, 14, and 18. **(14, 5)** gives the result that there exist at least two orthogonal Latin squares of orders $12t + 14$, $12t + 18$ and $12t + 22$. Since Euler's conjecture is false **(5)** for all numbers of the form $4t + 2$ which are $\leqslant 74$, excepting for 2 and 6, another proof for the falsity of the conjecture for all numbers $> 10$ could be given if it could be shown that a resolvable solution $F_2(12t + 4)$ exists for all $t > 5$.

## References


1. R. C. Bose, *On the construction of balanced incomplete block designs*, Ann. Eugen., *9* (1939), 353–399.
2. R. C. Bose and K. A. Bush, *Orthogonal arrays of strength two and three*, Ann. Math. Stat., *23* (1952), 508–524.
3. R. C. Bose and S. S. Shrikhande, *On the falsity of Euler's conjecture about the nonexistence of two orthogonal Latin squares of order* $4t + 2$, Proc. N. A. S., *45* (1959), 734–737.
4. ——— *On the construction of sets of pairwise orthogonal Latin squares and the falsity of a conjecture of Euler*, Trans. Amer. Math. Soc., to appear.
5. R. C. Bose, S. S. Shrikhande, and E. T. Parker, *Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture*, Can. J. Math., *12* (1960), 189–203.
6. K. A. Bush, *Orthogonal arrays of index unity*, Ann. Math. Stat., *23* (1952), 426–434.
7. R. A. Fisher, *An examination of the different possible solutions of a problem in incomplete blocks*, Ann. Eugen. London, *10* (1940), 52–75.
8. H. O. Hartley, S. S. Shrikhande, and W. B. Taylor, *A note on incomplete block designs with row balance*, Ann. Math. Stat., *24* (1953), 123–128.
9. C. G. Jacobi, *Canon Arithmeticus*, Akademie-verlag (Berlin, 1956).
10. H. F. MacNeish, *Euler's squares*, Ann. Math., Series 2, *23* (1922), 221–227.
11. H. B. Mann, *Analysis and design of experiments* (Dover, 1949).
12. E. Netto, *Lehrbuch der combinatorik* (Chelsea Publishing Co., New York).

**13.** E. T. Parker, *Construction of some sets of mutually orthogonal Latin squares*, Proc. Amer. Math. Soc., to appear.
**14.** ——— *Orthogonal Latin squares*, Proc. N. A. S., *45* (1959), 859–862.
**15.** R. L. Plackett and J. P. Burman, *The design of optimum multi-factorial experiments*, Biometrika, *33* (1943–6), 305–325.
**16.** C. R. Rao, *Factorial experiments derivable from combinatorial arrangements of arrays*, J. Royal Stat. Soc., Suppl. *9* (1947), 128–139.
**17.** C. A. B. Smith and H. O. Hartley, *The construction of Youden squares*, J. Royal Stat. Soc., Series B, *10* (1948), 262–263.
**18.** F. Yates, *Incomplete randomised blocks*, Ann. Eugen. London, *8* (1936), 121–140.

*University of North Carolina*
*and*
*Case Institute of Technology*

# FURTHER RESULTS ON THE CONSTRUCTION OF MUTUALLY ORTHOGONAL LATIN SQUARES AND THE FALSITY OF EULER'S CONJECTURE

R. C. BOSE, S. S. SHRIKHANDE, AND E. T. PARKER

**1. Introduction.** If

$$p_1^{n_1} p_2^{n_2} \ldots p_u^{n_u}$$

is the prime power decomposition of an integer $v$, and we define the arithmetic function $n(v)$ by

$$n(v) = \min(p_1^{n_1}, p_2^{n_2}, \ldots, p_u^{n_u}) - 1,$$

then it is known, MacNeish **(10)** and Mann **(11)**, that there exists a set of at least $n(v)$ mutually orthogonal Latin squares (m.o.l.s.) of order $v$. We shall denote by $N(v)$ the maximum possible number of mutually orthogonal Latin squares of order $v$. Then the Mann-MacNeish theorem can be stated as

$$N(v) \geqslant n(v).$$

MacNeish conjectured that the actual value of $N(v)$ is $n(v)$. This conjecture seemed plausible as it implied the correctness of Euler's conjecture **(8**, p. 383, § 144**)** about the non-existence of two orthogonal Latin squares of order $v = 4t + 2$ (since $n(v) = 1$ in this case), and also the well-known result

$$N(v) = n(v) = p^m - 1 \text{ when } v = p^m \text{ and } p \text{ is a prime.}$$ $n \geq 3$.

MacNeish's conjecture was disproved by Parker **(12)** who showed that in certain cases $N(v) > n(v)$ by proving that if there exists a balanced incomplete block (BIB) design with $v$ treatments, $\lambda = 1$, and block size $k$ which is a prime power then $N(v) \geqslant k - 2$, and that this result can be improved to $N(v) \geqslant k - 1$, when the design is symmetric and cyclic.

Parker's result though it did not disprove Euler's conjecture threw serious doubts on its correctness. Bose and Shrikhande **(4)** were able to obtain a counter example by using a general class of designs, viz., the pairwise balanced designs of index unity. They showed **(6)** that Euler's conjecture is false for an infinity of values of $v \geqslant 22$, and obtained improved lower bounds for $N(v)$ for a large class of values of $v$.

By using the method of differences Parker **(13)** showed that $N(v) \geqslant 2$ for

Received August 30, 1959. This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. AF 49(638)-213. Reproduction in whole or in part is permitted for any purpose of the United States Government.

$v = \frac{1}{2}(3q - 1)$, where $q$ is a prime power $\equiv 3 \pmod 4$. This includes the case $v = 10$.

In the present paper (i) the main theorem of **(6)** has been improved enabling us to obtain better bounds on $N(v)$, (ii) the method of differences has been used to show that $N(v) \geqslant 2$ when $v = 14$, 26, or $12t + 10$, and (iii) Euler's conjecture has been shown to be false for all $v = 4t + 2 > 6$.

**2. Definitions and notations.** We shall try to adhere as much as possible to the notation and definitions used in **(6)**.

A Latin square of order $v$ may be defined as an arrangement of $v$ symbols say, $1, 2, \ldots, v$ in a $v \times v$ square such that each symbol occurs exactly once in every row and once in every column. Two Latin squares are said to be orthogonal if, when they are superposed, each symbol of the first square occurs just once with each symbol of the second square. A set of mutually orthogonal Latin squares is a set of Latin squares any two of which are orthogonal.

An orthogonal array $(k^2, q + 1, k, 2)$ of size $k^2$, $q + 1$ constraints, $k$ levels and strength 2 is a $k^2 \times (q+1)$ matrix $A$ whose elements are $k$ symbols, such that every two-rowed submatrix of $A$ contains as a column vector every possible pair of symbols. It is well known **(7; 14)** that the existence of $q - 1$ mutually orthogonal $k \times k$ Latin squares implies the existence of an orthogonal array $(k^2, q + 1, k, 2)$ and conversely.

An arrangement of $v$ objects (called treatments) in $b$ sets (called blocks) will be called a pairwise balanced design of index unity and type $(v; k_1, k_2, \ldots, k_m)$ if each block contains either $k_1, k_2, \ldots,$ or $k_m$ treatments which are all distinct $(k_i \leqslant v, k_i \neq k_j)$, and every pair of distinct treatments occurs in exactly one block of the design. If the number of blocks containing $k_i$ treatments is $b_i$, then clearly

$$(2.1) \qquad b = \sum_{i=1}^{m} b_i, \qquad v(v - 1) = \sum_{i=1}^{m} b_i k_i (k_i - 1).$$

Consider a pairwise balanced design $(D)$ of index unity and type $(v; k_1, k_2, \ldots, k_m)$. The subdesign $(D_i)$ formed by the blocks of size $k_i$ will be called the $i$th equiblock component of $(D)$, $i = 1, 2, \ldots, m$.

A subset of blocks belonging to any equiblock component $(D_i)$ will be said to be of type I if every treatment occurs in the subset exactly $k_i$ times. The number of blocks in such a subset is clearly $v$. As noted by Levi using König's theorem on the decomposition of even regular graphs **(9**, pp. 4–6**)**, we can rearrange the treatments within the blocks of the subset in such a way that every treatment comes in each position exactly once. If the $v$ blocks of the subset are written out as columns, each treatment occurs exactly once in every row. When so written out the blocks will be said to be in the standard form. A subset of blocks belonging to $(D_i)$ will be said to be of type II if every treatment occurs in the subset exactly once. The component $(D_i)$ will be defined to be separable if the blocks can be divided into subsets of type I or

type II (both types may occur at the same time). The design $(D)$ is defined to be separable if each equiblock component is separable.

The set of equiblock components $(D_1), (D_2), \ldots, (D_l)$, $l < m$, will be said to be a clear set if the $\sum_{i=1}^{l} b_i$ blocks comprising $(D_1), (D_2), \ldots, (D_l)$ are disjoint, that is, no two blocks contain a common treatment. Clearly a necessary condition for this is

$$\sum_{i=1}^{l} b_i k_i \leqslant v.$$

We shall have occasion to use the following Lemmas proved in **(6)**.

LEMMA 1. *If* $v = v_1 v_2 \ldots v_u$ *then* $N(v) \geqslant \min(N(v_1), N(v_2), \ldots, N(v_u))$.

LEMMA 2. *Suppose there exists a set* $\sum$ *of* $q - 1$ m.o.l.s. *of order* $k$, *then we can construct a* $q \times k(k-1)$ *matrix* $P$, *whose elements are the symbols* $1, 2, \ldots, k$ *and such that* (i) *any ordered pair*

$$\binom{i}{j},\ i \neq j$$

*occurs as a column exactly once in any two-rowed submatrix of* $P$, (ii) $P$ *can be subdivided into* $k - 1$ *submatrices* $P_1, P_2, \ldots, P_{k-1}$ *of order* $q \times k$ *such that in each row of* $P_c$, $1 \leqslant c \leqslant k - 1$, *each of the symbols* $1, 2, \ldots, k$ *occurs exactly once.*

Let $\delta$ be a $k \times 1$ column vector, then following the notation used in **(6)**, we shall denote by $P(\delta)$ the $q \times k(k-1)$ matrix obtained from $P$ on replacing the symbol $i$ by the element occurring in the $i$th position in $\delta$. A similar meaning will be assigned to $P_i(\delta)$ and $\pi_{cj}(\delta)$ where $\pi_{cj}$ denotes the $j$th column of $P_c$. If $D$ is a $k \times b$ matrix defined by

$$D = [\delta_1, \delta_2, \ldots, \delta_b]$$

where $\delta_j$ is a $k \times 1$ column vector, then we define $P(D)$ and $P_i(D)$ by

$$P(D) = [P(\delta_1), P(\delta_2), \ldots, P(\delta_b)]$$
$$P_i(D) = [P_i(\delta_1), P_i(\delta_2), \ldots, P_i(\delta_b)].$$

**3. Main theorem.** The theorem proved in this section is an improvement of the main theorem of **(6)**.

THEOREM 1. *Let there exist a pairwise balanced design* $(D)$ *of index unity and type* $(v; k_1, k_2, \ldots, k_m)$ *such that the set of equiblock components* $(D_1), (D_2), \ldots, (D_l)$, $l < m$, *is a clear set. If there exist* $q_i - 1$ *mutually orthogonal Latin squares of order* $k_i$ *and if*

$$q^* = \min(q_1 + 1, \ldots, q_l + 1, q_{l+1}, \ldots, q_m),$$

*then there exist at least* $q^* - 2$ *mutually orthogonal Latin squares of order* $v$.

*Proof.* Let us define

$$q^{(1)} = \min(q_1 + 1, q_2 + 1, \ldots, q_l + 1)$$

and

$$q^{(2)} = \min(q_{l+1}, q_{l+2}, \ldots, q_m).$$

Then

$$q^* = \min(q^{(1)}, q^{(2)}).$$

Let

$$\delta_{i1}, \delta_{i2}, \ldots, \delta_{ib_i}$$

be the blocks of the equiblock component $(D_i)$ written out as columns $(i \leq l)$. By hypothesis there exist $q_i - 1$ mutually orthogonal Latin squares of order $k_i$. Hence we can construct an orthogonal array $A_{ij}$ with $q_i + 1$ rows and $k_i^2$ columns, whose symbols are the treatments occurring in $\delta_{ij}$. Let

$$A_i = [A_{i1}, A_{i2}, \ldots, A_{ib_i}].$$

Let $\Delta_i$ be the $q^* \times b_i k_i^2$ matrix obtained from $A_i$ by retaining only the first $q^*$ rows, and let

$$\Delta^{(1)} = [\Delta_1, \Delta_2, \ldots, \Delta_l].$$

Then $\Delta^{(1)}$ has $q^*$ rows and $\sum b_i k_i^2$ columns. Clearly $\Delta^{(1)}$ has the property that if $t_c$ and $t_d$ are any two treatments identical or distinct contained in any block of $(D_1), (D_2), \ldots,$ or $(D_l)$, then the ordered pair $t_c, t_d$ occurs as a column exactly once in any two-rowed submatrix of $\Delta^{(1)}$.

Let $\Delta_u$ be the matrix obtained from $P_u(D_u)$ by retaining only the first $q^*$ rows, $u = l + 1, \ldots, m$. Then

$$\Delta^{(2)} = [\Delta_{l+1}, \Delta_{l+2}, \ldots, \Delta_m]$$

has the property that if $t_a$ and $t_b$ are any two distinct treatments contained in any block of $(D_{l+1}), \ldots, (D_m)$ then the ordered pair $t_a, t_b$ occurs exactly once in any two-rowed submatrix of $\Delta^{(2)}$. The number of columns in $\Delta^{(2)}$ is

$$\sum_{u=l+1}^{m} b_u k_u (k_u - 1).$$

Again let $\Delta^{(3)}$ be the $q^* \times v_2$ matrix whose $n$th column contains in every position the treatment $t_n$, where $t_n$ is any one of the

$$v_2 = v - \sum_{i=1}^{l} b_i k_i$$

treatments not contained in $(D_1), (D_2), \ldots,$ or $(D_l)$. Then $[\Delta^{(1)}, \Delta^{(2)}, \Delta^{(3)}]$ is an orthogonal array $(v^2, q^*, v, 2)$, and using any two rows for co-ordinatization we get $q^* - 2$ mutually orthogonal Latin squares of order $v$.

**4. Use of BIB designs.** A balanced incomplete block (BIB) design with parameters $v$, $b$, $r$, $k$, $\lambda$ is an arrangement of $v$ objects or treatments into $b$ sets or blocks such that (i) each block contains $k < v$ different treatments,

(ii) each treatment occurs in $r$ different blocks, and (iii) each pair of treatments occurs together in exactly $\lambda$ blocks. The parameters satisfy the relations

$$\lambda(v-1) = r(k-1), \quad bk = vr, \quad b \geqslant v.$$

These conditions are necessary but not sufficient for the existence of a BIB design. BIB designs were first introduced into statistical studies by Yates **(15)**, but occur in earlier literature in connection with various combinatorial problems. Subsequent to Yates many authors have dealt with the problem of constructing these designs. Without attempting a complete bibliography we shall only refer to **(1)**. A BIB design is said to be symmetric if $v = b$, and in consequence $k = r$. A BIB design is said to be resolvable **(2)** if the blocks can be divided into sets, such that the blocks of a given set contain each treatment exactly once. A resolvable or a symmetric BIB design is evidently separable.

A BIB design with $\lambda = 1$ is clearly a pairwise balanced design of index unity and type $(v; k)$. We shall denote such a design by BIB $(v; k)$.

By omitting a single treatment from the design BIB $(v; k)$ we get a pairwise balanced design of index unity and type $(v-1; k, k-1)$ where the $r$ blocks of size $k-1$ form a clear set. Again if from a BIB $(v; k)$ we delete $x$ treatments belonging to the same block, $2 \leqslant x \leqslant k$, we get a pairwise balanced design of index unity and type $(v-x; k, k-1, k-x)$ where the equiblock component consisting of the single block of size $k-x$ is clear. Hence we have

THEOREM 2. *Existence of a BIB $(v; k)$ implies*

(i) $N(v-1) \geqslant \min(N(k), 1 + N(k-1)) - 1$,

(ii) $N(v-x) \geqslant \min(N(k), N(k-1), 1 + N(k-x)) - 1$, if $2 \leqslant x \leqslant k$.

Example (1). Consider the BIB design with parameters $v = b = s^2 + s + 1$, $r = k = s + 1$, $\lambda = 1$, with $s = 16$. Taking $x = 6, 8, 9$ respectively we get $N(267) \geqslant 10$, $N(265) \geqslant 8$, $N(264) \geqslant 7$, whereas $n(267) = 2$, $n(265) = 4$ and $n(264) = 2$.

Suppose we omit three treatments $\alpha_1, \alpha_2, \alpha_3$ not occurring in the same block of a BIB $(v; k)$, then we get a pairwise balanced design $(D)$ of index unity and type $(v-3; k, k-1, k-2)$. Since in the original BIB $(v; k)$ no two blocks can have more than one treatment in common, the three blocks of $(D)$ of size $k-2$ which have been obtained by deleting $(\alpha_1, \alpha_2)$, $(\alpha_2, \alpha_3)$, $(\alpha_1, \alpha_3)$ have obviously no treatment in common and form a clear equiblock component. Hence we get

THEOREM 3. *The existence of a* BIB $(v; k)$ *implies that*

$$N(v-3) \geqslant \min(N(k), N(k-1), 1 + N(k-2)) - 1.$$

Example (2). Consider the BIB $(v; k)$ designs **(1**, pp. 386–389**)** with $k = 5$ and $v = 21, 25, 41, 45, 61, 65, 85, 125$. It follows that there exist at least two m.o.l.s. of the following orders: 18, 22, 38, 42, 58, 62, 82, and 122.

Example (3). From the designs BIB (81; 9) and BIB (73; 9) we get $N(78) \geq 6$, $N(70) \geq 6$.

Example (4). From the design BIB (273; 17) we get $N(270) \geq 2$ since $N(15) \geq n(15) = 2$.

Suppose there exists a resolvable BIB design with parameters $v$, $b$, $r$, $k$, $\lambda = 1$. Let $1 < x \leq r$. To each block of the $i$th replication add a new treatment $\theta_i$, $i = 1, 2, \ldots, x$, and add a new block $\theta_1, \theta_2, \ldots, \theta_x$. We then get a pairwise balanced design of index unity and type $(v + x; k + 1, k, x)$ if $x < r$, and type $(v + x; k + 1, r)$ if $x = r$. The equiblock component formed by the new block is clear. When $x = r - 1$, the set of equiblock components consisting of the new block, and the blocks of the $r$th replication is a clear set. Again by adding a treatment $\theta$ to all the blocks of a single replication we get $(v + 1; k + 1, k)$. Hence we have

THEOREM 4A. *The existence of a resolvable* BIB $(v; k)$ *implies*

(i) $N(v + x) \geq \min(N(k), N(k + 1), 1 + N(x)) - 1$ if $1 < x \leq r - 2$,
(ii) $N(v + r - 1) \geq \min(1 + N(k), N(k + 1), 1 + N(r - 1)) - 1$,
(iii) $N(v + r) \geq \min(N(k + 1), 1 + N(r)) - 1$,
(iv) $N(v + 1) \geq \min(N(k), N(k + 1)) - 1$.

Example (5). Taking $x = 5$ in the BIB design with $v = 49$, $b = 56$, $r = 8$, $k = 7$, $\lambda = 1$, we have

$$N(54) \geq \min(N(7), N(8), 1 + N(5)) - 1 = 4.$$

Example (6). Using the resolvable BIB design $v = 21$, $b = 70$, $r = 10$, $k = 3$, $\lambda = 1$, (**5**, p. 171, Table III) we have from part (ii) of the theorem $N(30) \geq 2$.

Again suppose there exists a separable BIB $(v; k)$ in which the blocks can be divided into $n$ sets of type I. The number of replications is $r = kn$. For a symmetric design $n = 1$. Let the blocks be written out as columns, and let $(S_j)$ be the $j$th subset, the blocks being in the standard form ($j = 1, 2, \ldots, n$). Let us take $r$ new treatments $\theta_{ij}$, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n$. Let us define the $1 \times v$ row vector $\Theta_{ij} = (\theta_{ij}, \theta_{ij}, \ldots, \theta_{ij})$. Then we can denote by

$$\begin{pmatrix} S_j \\ \Theta_{ij} \end{pmatrix}$$

the result of adding $\theta_{ij}$ in the $(k + 1)$th position to each block of the $j$th subset.

Let $N(k + 1) = q^{(1)} - 1$. Then we can construct a $q^{(1)} \times (k + 1)k$ matrix $P^{(1)}$ with the properties (i) and (ii) of Lemma 2, where $P_1^{(1)}, P_2^{(1)}, \ldots, P_k^{(1)}$ are the submatrices referred to in part (ii). If $\delta_{ju}$ is the $u$th block of $(S_j)$, $u = 1, 2, \ldots, v$, then the corresponding block of

$$\begin{pmatrix} S_j \\ \Theta_{ij} \end{pmatrix}$$

is

$$\binom{\delta_{ju}}{\theta_{ij}}.$$

Consistent with our notation we can denote by

$$P_i^{(1)}\binom{\delta_{ju}}{\theta_{ij}}$$

the result of replacing the symbols $1, 2, \ldots, k, k+1$ in $P_i^{(1)}$ by treatments in the 1st, 2nd, ..., $(k+1)$th position in

$$\binom{\delta_{ju}}{\theta_{ij}}$$

and define

$$P_i^{(1)}\binom{S_j}{\Theta_{ij}} = \left[P_i^{(1)}\binom{\delta_{j1}}{\theta_{ij}}, \ldots, P_i^{(1)}\binom{\delta_{jv}}{\theta_{ij}}\right].$$

A pair of distinct treatments belonging to the original BIB design may be called a pure pair. Again a pair of treatments, one of which belongs to the original BIB design, and the other to the newly added treatments, may be called a mixed pair. Then

$$\Delta_1 = \left[\ldots, P_i^{(1)}\binom{S_j}{\Theta_{ij}}, \ldots\right], i = 1, 2, \ldots, k; j = 1, 2, \ldots, n$$

has the property that any two-rowed submatrix contains as a column each pure and each mixed ordered pair of treatments exactly once.

Again if $q^{(2)} - 1 = N(r)$, we can form an orthogonal array $\Delta_2 = (r^2, q^{(2)}+1, r, 2)$ whose symbols are the $r$ new treatments. Let

$$q = \min(q^{(1)}, q^{(2)} + 1)$$

and let $\Delta^{(1)}$ and $\Delta^{(2)}$ be obtained from $\Delta_1$ and $\Delta_2$ respectively by retaining the first $q$ rows only. Also let $\Delta^{(3)}$ be the $q \times v$ matrix whose $u$th column contains the $u$th treatment of the BIB design in each position. Then

$$\Delta = [\Delta^{(1)}, \Delta^{(2)}, \Delta^{(3)}]$$

is an orthogonal array of size $(v + r)^2$, $q$ constraints, $v + r$ levels and strength 2, and is therefore equivalent to a set of $q - 2$ mutually orthogonal Latin squares. Hence we have

THEOREM 4B. *If there exists a* BIB $(v; k)$ *with $r$ replications, in which the blocks can be subdivided into sets of type I, then*

$$N(v + r) \geqslant \min(N(k + 1), 1 + N(r)) - 1.$$

Example (7). The existence of symmetric BIB $(7; 3)$ and BIB $(57; 8)$ implies $N(10) \geqslant 2$ and $N(65) \geqslant 7$.

Example (8). There exists a BIB design **(1**, p. 383**)** with parameters $v = 25$, $b = 50$, $r = 8$, $k = 4$, $\lambda = 1$ for which the blocks can be separated into two subsets of type I. It follows that $N(33) \geqslant 3$.

Theorems 4A and 4B are special cases of the following more general theorem, the proof of which can be given on analogous lines.

THEOREM 4. *If there exists a separable* BIB $(v; k)$ *with* $n_1$ *subsets of type I and* $n_2$ *subsets of type II, so that the number of replications is* $r = kn_1 + n_2$, *then*

(i) $N(v + x) \geqslant \min(N(k), N(k + 1), 1 + N(x)) - 1$, if $x = kr_1 + r_2$; $r_1 \leqslant n_1$, $r_2 \leqslant n_2$; $1 < x < r - 1$.

(ii) $N(v + r - 1) \geqslant \min(1 + N(k), N(k + 1), 1 + N(r - 1)) - 1$ if $n_2 > 0$.

(iii) $N(v + r) \geqslant \min(N(k + 1), 1 + N(r)) - 1$.

(iv) $N(v + 1) \geqslant \min(N(k + 1), N(k)) - 1$ if $n_2 > 0$.

**5. Use of GD designs.** An arrangement of $v$ objects (treatments) in $b$ sets (blocks) each containing $k$ distinct treatments is said to be a group divisible (GD) design if the treatments can be divided into $l$ groups of $m$ treatments each, so that any two treatments belonging to the same group occur together in $\lambda_1$ blocks, and any two treatments from different groups occur together in $\lambda_2$ blocks. We will denote such a design by the notation GD $(v; k, m; \lambda_1, \lambda_2)$. The combinatorial properties of these designs have been studied in **(3)** where it has been shown that

$$v = lm, \quad bk = vr, \quad \lambda_1(m - 1) + \lambda_2 m(l - 1) = r(k - 1),$$

$r$ being the number of replications, that is, the number of times each treatment occurs in the design. It has also been shown that

$$P = rk - \lambda_2 v \geqslant 0, \qquad Q = r - \lambda_1 \geqslant 0.$$

The GD designs can be divided into three classes.

(i) Regular (R) characterized by $P > 0$, $Q > 0$.

(ii) Semi-regular (SR) characterized by $P = 0$, $Q > 0$.

(iii) Singular (S) characterized by $Q = 0$.

Methods of constructing these designs have been given in **(5)**. So far as the construction of mutually orthogonal Latin squares is concerned a special role is played by GD designs with $\lambda_1 = 0$, $\lambda_2 = 1$, which in our notation can be denoted by GD$(v; k, m; 0, 1)$. If, further, this design is regular we shall denote it by RGD$(v; k, m; 0, 1)$ and if it is semi-regular we shall denote it by SRGD$(v; k, m; 0, 1)$.

If to the $b$ blocks of the GD design with $\lambda_1 = 0$, $\lambda_2 = 1$, we add $l$ new blocks corresponding to the groups, we get a pairwise balanced design of index unity and type $(v; k, m)$. The blocks of size $m$ form a clear equiblock component. Hence we have

THEOREM 5. *If there exists a* GD$(v; k, m; 0, 1)$ *then*

$$N(v) \geqslant \min(N(k), 1 + N(m)) - 1.$$

COROLLARY. *$N(s^2 - 1) \geq N(s - 1)$, if $s$ is a prime power.*

This follows from the existence of a resolvable $\mathrm{GD}(s^2 - 1; s, s - 1; 0, 1)$.

THEOREM 6. *If there exists a* $\mathrm{GD}(v; k, m; 0, 1)$ *then*

$$N(v - 1) \geq \min(N(k), N(k - 1), 1 + N(m), 1 + N(m - 1)) - 1,$$

*and if the design is resolvable then*

$$N(v - 1) \geq \min(N(k), N(k - 1), N(m), N(m - 1)).$$

The first part follows from the fact that if we omit any particular treatment from the corresponding pairwise balanced design of index unity and type $(v; k, m; 0, 1)$ we get a design of the type $(v - 1; k, k - 1; m, m - 1)$, in which the equiblock components with blocks of sizes $m$ and $m - 1$ form a clear set. The second part has already been proved in **(6)** and is given here for completeness.

THEOREM 7. *Suppose there exists a resolvable* $\mathrm{GD}(v; k, m; 0, 1)$ *with $r$ replications, then*

(i) $N(v + 1) \geq \min(N(k), N(k + 1), 1 + N(m)) - 1$,

(ii) $N(v + x) \geq \min(N(k), N(k + 1), 1 + N(m), 1 + N(x)) - 1$ if $1 < x < r$,

(iii) (a) $N(v + r) \geq \min(N(k + 1), 1 + N(m), 1 + N(r)) - 1$,

(b) $N(v + r) \geq \min(N(k + 1), N(m + 1), 1 + N(k), 1 + N(r)) - 1$,

(iv) $N(v + r + 1) \geq \min(N(k + 1), N(m + 1), 1 + N(r + 1)) - 1$,

*where in part* (iii) *we choose whichever lower bound is better for $N(v + r)$.*

To prove part (i) we add a new treatment $\theta_1$ to each block of one replication. To prove part (ii) we add a new treatment $\theta_i$ to each block of the $i$th replication, $i = 1, 2, \ldots, x$, and take a new block $(\theta_1, \theta_2, \ldots, \theta_x)$. For the first part we note that the equiblock component given by the groups forms a clear set. For the second part we note that the set of equiblock components given by the groups and the new block is a clear set. To prove part (iii) (a) we add a new treatment $\theta_i$ to each block of the $i$th replication, $i = 1, 2, \ldots, r$, and a new block $(\theta_1, \theta_2, \ldots, \theta_r)$. To prove part (iii) (b) we add a new treatment $\theta_i$ to each block of the $i$th replication for the first $r - 1$ replications, and a new treatment $\theta_0$ to each of the groups, and add a new block $(\theta_0, \theta_1, \ldots, \theta_{r-1})$. We note in this case that the set of equiblock components given by the $r$th replication, and the newly added block is a clear set. To prove (iv) we add one new treatment to the blocks of each replication, one new treatment to each of the blocks corresponding to the groups, and take a block containing the new treatments.

The group designs most useful to us are the semi-regular group divisible designs with $\lambda_1 = 0$, $\lambda_2 = 1$. For such a design the number of replications $r$ is equal to the group size $m$, and $v = km$ In the notation used in **(6)** such a design is denoted by SRGD $(km; k, m; 0, 1)$. It is known **(5)** that for an

SRGD design each block contains the same number of treatments from each group. We shall now prove

LEMMA 3. *There exists a resolvable* SRGD $(km; k, m; 0, 1)$ *if* $k \leqslant N(m) + 1$. *For this design the block size is* $k$, $r = m$, *and* $b = m^2$.

Applying Lemma 2 we find that there exists a matrix $P$ with $N(m) + 1$ rows and $m(m-1)$ columns such that in every two-rowed submatrix of $P$ every ordered pair of distinct symbols occurs exactly once, and it can be subdivided into $m - 1$ parts such that in each row of every part every symbol occurs once. Let $P_k$ be the matrix obtained from $P$ by retaining only $k$ rows. Let $E_k$ be a $k \times m$ matrix such that the $i$th column contains the $i$th symbol in each position. Let $\Delta_k = [E_k, P_k]$. Now let us consider $km$ treatments $t_1, t_2, \ldots, t_{km}$. Let the symbols $1, 2, \ldots, m$ in the $j$th row of $\Delta_k$ be replaced by

$$t_{\alpha+1}, t_{\alpha+2}, \ldots, t_{\alpha+m} \quad \text{where} \quad \alpha = (j-1)m.$$

This gives an SRGD with the required properties. The blocks are given by the columns. The replications consist of $E_k$ and subdivisions of $P_k$.

Combining Lemma 3 and Theorem 7 we have

THEOREM 8. *If* $k \leqslant N(m) + 1$, *then*

(i) $N(km + 1) \geqslant \min(N(k), N(k+1), 1 + N(m)) - 1$,

(ii) $N(km + x) \geqslant \min(N(k), N(k+1), 1 + N(m), 1 + N(x)) - 1$ if $1 < x < m$.

Example (9). Taking $k$, $m$, and $x$ as shown we derive the lower bound for $N(km + x)$, noting that $N(24) \geqslant 3$ from Table 1 of **(6)** and $N(10) \geqslant 2$ from Example (7), Theorem 4B.

| | | |
|---|---|---|
| (i) | $k = 7,\ m = 11,\ x = 5;$ | $N(82) \geqslant 4,$ |
| (ii) | $k = 8,\ m = 11,\ x = 7;$ | $N(95) \geqslant 6,$ |
| (iii) | $k = 7,\ m = 19,\ x = 5;$ | $N(138) \geqslant 4,$ |
| (iv) | $k = 7,\ m = 8,\ x = 4;$ | $N(60) \geqslant 3,$ |
| (v) | $k = 4,\ m = 24,\ x = 10;$ | $N(106) \geqslant 2,$ |
| (vi) | $k = 8,\ m = 13,\ x = 7;$ | $N(111) \geqslant 6,$ |
| (vii) | $k = 4,\ m = 27,\ x = 10;$ | $N(118) \geqslant 2,$ |
| (viii) | $k = 7,\ m = 16,\ x = 10;$ | $N(122) \geqslant 2,$ |
| (ix) | $k = 7,\ m = 17,\ x = 5;$ | $N(124) \geqslant 4.$ |

**6. Use of the method of differences.** Let $0, 1, 2, \ldots, n-1$ be the elements of the ring $R$ of residue classes (mod $n$). We shall consider matrices whose elements belong either to $R$ or to the set $X$ of $m$ indefinites $x_1, x_2, \ldots, x_m$. We shall say that the difference associated with the ordered pair $\binom{i}{j}$, where $i$ and $j$ belong to $R$ is $c$ where $i - j \equiv c \pmod n$, $0 \leqslant c < n$. Conversely to each element $c$ of $R$ there correspond $n$ ordered pairs which have $c$ as their associated difference. If $\binom{i}{j}$ is one of these pairs then the other pairs are $\binom{i+\theta}{j+\theta}$ where $\theta = 0, 1, 2, \ldots, n-1$, and $i + \theta$ and $j + \theta$ are reduced (mod $n$).

The ordered pair $\binom{i}{j}$ both members of which belong to $R$ will be called an $R$-pair. A pair $\binom{i}{x_j}$ where $i$ belongs to $R$ and $x_j$ to $X$ is called an $RX$-pair and the difference associated with it is defined to be $x_j$. If $\theta$ is any element of $R$ we shall formally define $x_j + \theta = x_j$. With this definition, corresponding to any indefinite $x_j$, there are $n$ $RX$ pairs, the difference associated with each of which is $x_j$. If $\binom{i}{x_j}$ is one of these pairs then the other pairs are $\binom{i+\theta}{x_j}$ where $\theta = 0, 1, \ldots, n-1$. These pairs are of course all the pairs $\binom{i}{x_j}$, $i = 0, 1, \ldots, n-1$ in some order or other. We may similarly define $XR$ pairs. The difference associated with the $XR$ pair $\binom{x_i}{j}$ is $x_i$.

We shall now prove the following theorem:

THEOREM 9. *If $m$ is odd there exist at least two orthogonal Latin squares of order $3m+1$. Taking $m = 4t+3$ this implies the existence of a pair of orthogonal Latin squares for all orders $12t+10$.*

Consider the $4 \times 4m$ matrix $A_0$ given below (to exhibit the structure of $A_0$ it is divided into 4 parts), whose elements belong to $R$ the ring of residue classes $\bmod(2m+1)$ or $X$ the set of indefinites $x_1, x_2, \ldots, x_m$.

$$A_0 = \left[\begin{array}{llllllllllll}
0 & 0 & \ldots 0 & 1 & 2 & \ldots m & 2m & 2m-1 & \ldots m+1 & x_1 & x_2 & \ldots x_m \\
1 & 2 & \ldots m & 0 & 0 & \ldots 0 & x_1 & x_2 & \ldots x_m & 2m & 2m-1 & \ldots m+1 \\
2m & 2m-1 & \ldots m+1 & x_1 & x_2 & \ldots x_m & 0 & 0 & \ldots 0 & 1 & 2 & \ldots m \\
x_1 & x_2 & \ldots x_m & 2m & 2m-1 & \ldots m+1 & 1 & 2 & \ldots m & 0 & 0 & \ldots 0
\end{array}\right].$$

We note that of the $4m$ pairs occurring as columns in any two-rowed submatrix of $A_0$, $2m$ are $R$-pairs, the differences associated with which are all the non-null elements of $R$, $m$ are $RX$-pairs the differences associated with which are all the elements of $X$ and the same is true of $XR$ pairs. Let $A_\theta$ be the matrix derived from $A_0$ by adding $\theta$, $0 < \theta < 2m$, to every element of $A_0$ and reducing $\bmod(2m+1)$; $x_i + \theta$ being considered as $x_i$. Let

$$A = [A_0, A_1, \ldots, A_{2m}].$$

Then it is evident that in any two-rowed submatrix of $A$, any $R$-pair formed by two distinct elements of $R$, or any $RX$ or $XR$-pair occurs exactly once. Let $A^*$ be an orthogonal array $[m^2, 4, m, 2]$ corresponding to two orthogonal Latin squares formed by the symbols $x_1, x_2, \ldots, x_m$, and let $E$ be a $4 \times (2m+1)$ matrix whose $i$th column contains $i$ in each place $(0 < i < 2m)$. Then

$$\Delta = [E, A, A^*]$$

is an orthogonal array $[(3m+1)^2, 4, 3m+1, 2]$, which proves the result.

Example (10). Taking $t = 0, 1, 2, 3, 4, 8, 9,$ and $12$ respectively it follows that $N(v) > 2$ for $v = 10, 22, 34, 46, 58, 106, 118,$ and $154$.

Two superposed $10 \times 10$ orthogonal squares obtained by this method are exhibited below. The symbols $x_1, x_2, x_3$ have been replaced by 7, 8, 9.

A PAIR OF $10 \times 10$ ORTHOGONAL SQUARES

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 00 | 67 | 58 | 49 | 91 | 83 | 75 | 12 | 24 | 36 |
| 76 | 11 | 07 | 68 | 59 | 92 | 84 | 23 | 35 | 40 |
| 85 | 70 | 22 | 17 | 08 | 69 | 93 | 34 | 46 | 51 |
| 94 | 86 | 71 | 33 | 27 | 18 | 09 | 45 | 50 | 62 |
| 19 | 95 | 80 | 72 | 44 | 37 | 28 | 56 | 61 | 03 |
| 38 | 29 | 96 | 81 | 73 | 55 | 47 | 60 | 02 | 14 |
| 57 | 48 | 39 | 90 | 82 | 74 | 66 | 01 | 13 | 25 |
| 21 | 32 | 43 | 54 | 65 | 06 | 10 | 77 | 88 | 99 |
| 42 | 53 | 64 | 05 | 16 | 20 | 31 | 89 | 97 | 78 |
| 63 | 04 | 15 | 26 | 30 | 41 | 52 | 98 | 79 | 87 |

We shall now give two special examples of the use of the method of differences.

Example (11). Consider the matrix

$$P_0 = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ 1 & 0 & 0 & 0 \\ 4 & 4 & 6 & 9 \\ 6 & 1 & 2 & 8 \end{bmatrix}$$

whose elements belong to the ring $R$ of residue classes (mod 11) and the set $X$ of indefinites $x_1, x_2, x_3$. Let $P_1, P_2, P_3$ be obtained from $P_0$ by cyclic permutation of the rows, and let

$$A_0 = [P_0, P_1, P_2, P_3].$$

Then it is easy to verify that each two-rowed submatrix of $A_0$ contains as columns 10 $R$-pairs, the differences associated with which are all the non-null elements of $R$; 3 $RX$-pairs the differences associated with which are the 3 elements of $X$, and 3 $XR$ pairs for which the same is true. Let $A_\theta$ be the matrix obtained from $A_0$ by adding $\theta$ to elements of $A_0$, where $\theta$ belongs to $R$. Then

$$A = [A_0, A_1, \ldots, A_{10}]$$

is a matrix such that any two-rowed submatrix contains as a column every $R$-pair consisting of distinct elements of $R$, and every $RX$ and $XR$-pair, exactly once. If $A^*$ is the orthogonal array of strength 2 and 4 constraints with the symbols $x_1, x_2, x_3$ and $E$ is the $4 \times 11$ matrix for which the $i$th column contains $i$ in every place $(i = 0, 1, \ldots, 10)$ then

$$\Delta = [E, A, A^*]$$

is an orthogonal array $[14^2, 4, 14, 2]$ from which a pair of orthogonal Latin squares of order 14 can be constructed.

Example (12). Similarly by starting with the matrix

$$P_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 \\ 3 & 6 & 2 & 1 & 0 & 0 & 0 \\ 8 & 20 & 12 & 16 & 20 & 17 & 8 \\ 12 & 16 & 7 & 2 & 19 & 6 & 21 \end{bmatrix}$$

whose elements belong to the ring $R$ of residue classes (mod 23) and the set $X$ of indefinites $x_1, x_2, x_3$, we can construct two mutually orthogonal Latin squares of order 26.

**7. Improved lower bounds for $N(v)$, $v \leqslant 154$.** We give here those values of $v \leqslant 154$ for which the lower bound of $N(v)$ can be improved over the bound given in Table I of **(6)**.

TABLE I

| $v$ | $n(v)$ | l.b. for $N(v)$ | Remarks |
|---|---|---|---|
| 10 | 1 | 2 | Ex. (7), Th. 4B or Ex. (10), Th. 9 |
| 14 | 1 | 2 | Ex. (11) |
| 18 | 1 | 2 | Ex. (2), Th. 3 |
| 26 | 1 | 2 | Ex. (12) |
| 30 | 1 | 2 | Ex. (6), Th. 4A |
| 33 | 2 | 3 | Ex. (8), Th. 4B |
| 34 | 1 | 2 | Ex. (10), Th. 9 |
| 38 | 1 | 2 | Ex. (2), Th. 3 |
| 42 | 1 | 2 | Ex. (2), Th. 3 |
| 46 | 1 | 2 | Ex. (10), Th. 9 |
| 54 | 1 | 4 | Ex. (5), Th. 4A |
| 60 | 2 | 3 | Ex. (9) (iv), Th. 8 |
| 62 | 1 | 2 | Ex. (2), Th. 3 |
| 65 | 4 | 7 | Ex. (7), Th. 4B |
| 70 | 1 | 6 | Ex. (3), Th. 3 |
| 78 | 1 | 6 | Ex. (3), Th. 3 |
| 82 | 1 | 4 | Ex. (9) (i), Th. 8 |
| 90 | 1 | 2 | $90 = 10 \times 9$ and Lemma 1 |
| 95 | 4 | 6 | Ex. (9) (ii), Th. 8 |
| 106 | 1 | 2 | Ex. (10), Th. 9, or Ex. (9) (v), Th. 8 |
| 111 | 2 | 6 | Ex. (9) (vi), Th. 8 |
| 114 | 1 | 2 | $114 = 38 \times 3$ and Lemma 1 |
| 118 | 1 | 2 | Ex. (10), Th. 9, or Ex. (9) (vii), Th. 8 |
| 122 | 1 | 2 | Ex. (2), Th. 3, or Ex. (9) (viii), Th. 8 |
| 124 | 3 | 4 | Ex. (9) (ix), Th. 8 |
| 138 | 1 | 4 | Ex. (9) (iii), Th. 8 |
| 154 | 1 | 2 | Ex. (10), Th. 9, or $154 = 22 \times 7$ and Lemma 1 |

**8. The existence of at least two orthogonal Latin squares of order** $v > 6$. If $v$ is divisible by 4 or if $v$ is odd then $N(v) \geqslant n(v) \geqslant 2$. Hence we need only consider numbers for which $v \equiv 2 \pmod 4$. We shall first prove

LEMMA 4. $N(v) \geqslant 2$ *if* $6 < v \leqslant 726$.

This result has already been checked in Table I of **(6)**, supplemented by the improvements noted in Table I of the last section, up to $v = 154$.

Any integer $v$ lying in the closed interval $I_i = (a_i, b_i)$ shown in column (2) of Table II can be expressed in the form

$$v = 4m_i + x_i, \qquad 10 \leqslant x_i \leqslant c_i$$

where $m_i$ and $c_i$ are given in columns (3) and (4), since $a_i = 4m_i + 10$, $b_i = 4m_i + c_i$.

TABLE II

| $i$ | Interval $I_i = (a_i, b_i)$ | $m_i$ | $c_i$ |
|---|---|---|---|
| 1 | (158, 182) | 37 | 34 |
| 2 | (186, 218) | 44 | 42 |
| 3 | (222, 262) | 53 | 50 |
| 4 | (266, 310) | 64 | 54 |
| 5 | (314, 374) | 76 | 70 |
| 6 | (378, 454) | 92 | 86 |
| 7 | (458, 550) | 112 | 102 |
| 8 | (554, 662) | 136 | 118 |
| 9 | (666, 726) | 164 | 70 |

It is readily verified that $N(m_i) \geqslant n(m_i) \geqslant 3$. Again $N(x_i) \geqslant 2$ since $10 \leqslant x_i \leqslant c_i < 154$. If we take $k = 4$ in part (ii) of Theorem 8, the conditions $k \leqslant N(m_i) + 1$ and $1 < x_i < m_i$ are obviously satisfied. Hence if $v$ lies in any of the closed intervals $I_i\,(i = 1, 2, \ldots, 9)$, $N(v) \geqslant 2$. The Lemma follows by noting that any $v \equiv 2 \pmod 4$ and satisfying $154 < v \leqslant 726$ lies in one of the closed intervals $I_i$.

THEOREM 10. *There exist at least two orthogonal Latin squares of any order* $v > 6$.

It is sufficient to prove the theorem for numbers $v \equiv 2 \pmod 4$, $v \geqslant 730$. If $v$ satisfies these conditions we can write

$$v - 10 = 144g + 4u, \qquad g \geqslant 5, \qquad 0 \leqslant u \leqslant 35$$

therefore

$$v = 4(36g) + 4u + 10.$$

Since the least factor in the prime power decomposition of $36g$ is necessarily greater than or equal to 4, $N(36g) \geqslant n(36g) \geqslant 3$. If in Theorem 8 part (ii)

we take $k = 4$, $m = 36g$, $x = 4u + 10$, then $k < 1 + N(m)$. Also $10 < x < 150$, $m > 180$. Hence $1 < x < m$, and $N(x) > 2$. It follows that $N(v) > 2$.

The question raised in the concluding remarks of **(6)** is thus completely answered. If a positive integer $v > 2$ is called Eulerian if two orthogonal Latin squares of order $v$ do not exist, then 6 is the only Eulerian number.

## References


1. R. C. Bose, *On the construction of balanced incomplete block designs*, Ann. Eugen. London, *9* (1939), 353-399.
2. ——— *A note on the resolvability of balanced incomplete block designs*, Sankhyā, *6* (1942), 105–110.
3. R. C. Bose and W. S. Connor, *Combinatorial properties of group divisible incomplete block designs*, Ann. Math. Stat., *23* (1952), 367–383.
4. R. C. Bose and S. S. Shrikhande, *On the falsity of Euler's conjecture about the non-existence of two orthogonal Latin squares of order 4t + 2*, Proc. Nat. Acad. Sci. U.S.A., *45* (1959), 734–737.
5. R. C. Bose, S. S. Shrikhande, and K. Bhattacharya, *On the construction of group divisible incomplete block designs*, Ann. Math. Stat., *24* (1953), 167–195.
6. ——— *On the construction of pairwise orthogonal Latin squares and the falsity of a conjecture of Euler*, U. N. C., Institute of Statistics, Mimeo. series no. 222. To be published in Trans. Amer. Math. Soc.
7. K. A. Bush, *Orthogonal arrays of index unity*, Ann. Math. Stat., *23* (1952), 426–434.
8. L. Euler, *Recherches sur une nouvelle espece des quarres magiques*, Verh. zeeuwsch Genoot. Weten. Vliss., *9* (1782), 85–239.
9. F. W. Levi, *Finite geometrical systems* (University of Calcutta, 1942).
10. H. F. MacNeish, *Euler squares*, Ann. Math., *23* (1922), 221–227.
11. H. B. Mann, *The construction of orthogonal Latin squares*, Ann. Math. Stat., *13* (1942), 418–423.
12. E. T. Parker, *Construction of some sets of pairwise orthogonal Latin squares*, Amer. Math. Soc. Notices *5* (1958), 815 (abstract). (To be published in Proc. Amer. Math. Soc. under the title: *Construction of some sets of mutually orthogonal Latin squares*.)
13. ——— *Orthogonal Latin squares*, Proc. Nat. Acad. Sci. U.S.A., *45* (1959), 859–862.
14. C. R. Rao, *Factorial experiments derivable from combinatorial arrangements of arrays*, J. Roy. Stat. Soc. Suppl., *9* (1947), 128–139.
15. F. Yates, *Incomplete randomised blocks*, Ann. Eugen. London, *7* (1936), 121–140.


*University of North Carolina,*
*Case Institute of Technology*
*and*
*Remington Rand Univac, St. Paul, Minnesota*

# ON THE MAXIMAL NUMBER OF PAIRWISE ORTHOGONAL LATIN SQUARES OF A GIVEN ORDER

S. CHOWLA, P. ERDÖS, AND E. G. STRAUS

**1. Introduction.** In the preceding paper Bose, Shrikhande, and Parker give their important discovery of the disproof of Euler's conjecture on Latin squares. In this paper we show that their results can be strengthened to imply that $N(n)$, the maximal number of pairwise orthogonal Latin squares of order $n$, tends to infinity with $n$. In fact there exists a positive constant $c$, such that $N(n) > n^c$ for all sufficiently large $n$.

Our proof involves no new combinatorial insights, but is based entirely on a number-theoretical investigation of the following inequality due to Bose and Shrikhande.

THEOREM A. *If* $k \leqslant N(m) + 1$ *and* $1 < u < m$ *then* $N(km + u) \geqslant \min\{N(k), N(k + 1), 1 + N(m), 1 + N(u)\} - 1$.

The only other results on Latin squares which we need are due to H. F. MacNeish.

THEOREM B. (1) $N(ab) \geqslant \min\{N(a), N(b)\}$.
(2) $N(q) = q - 1$ *if* $q$ *is the power of a prime.*

In § 2 we give a proof of the fact that $N(n)$ tends to infinity, using only the most elementary tools. In § 3 we use Brun's method to obtain quantitative results on the lower bound of $N(n)$. Finally, in § 4, we discuss the theoretical limitations on the results that can be derived from Theorem A.

**2. Proof that**

$$\lim_{n\to\infty} N(n) = \infty.$$

Let $x$ be an arbitrarily large positive integer. Let

$$k + 1 = \prod_{p<x} p^x \qquad (p \text{ prime}). \tag{1}$$

Then by Theorem B we have

$$N(k + 1) \geqslant 2^x - 1 \geqslant x \tag{2}$$

and

Received August 8, 1959. This paper was written while the authors were members of the Number Theory Institute (Summer, 1959) in Boulder, Colorado. The authors wish to acknowledge with gratitude the opportunity for collaboration given them by this Institute.

$$N(k) > x \tag{3}$$

since all prime factors of $k$ are greater than $x$. Now set

$$m_1 = k^k \prod_{\substack{q \nmid n \\ q < x}} q^k \qquad (q \text{ prime}). \tag{4}$$

Note that while $m_1$ is defined in terms of $n$, it has an upper bound which depends on $x$ alone. If $n$ is sufficiently large then the interval $(n/(k+1)m_1, (n-1)/km_1)$ contains a number $m_2$ such that

$$m_2 \equiv 1 \pmod{k!}. \tag{5}$$

Thus the least prime factor of $m_2$ is greater than $k$.

If we set $m = m_1 m_2$ then from Theorem B and equations (4), (5) we obtain

$$\begin{aligned} N(m) &> \min\{N(m_1), N(m_2)\} > \min\{2^k - 1, k\}. \\ &> k. \end{aligned} \tag{6}$$

Thus the first condition of Theorem A is satisfied. Finally we set $u = n - km$. Since we had chosen $n/(k+1)m_1 < m_2 < (n-1)/km_1$ we have $km + 1 < n < (k+1)m$, so that

$$1 < u < m \tag{7}$$

which satisfies the second condition of Theorem A. From (1), (4), and (5) we see that $n$ and $km$ are incongruent module any prime less than $x$ and therefore $u$ has no divisors less than $x$. Thus

$$N(u) > x. \tag{8}$$

Combining (2), (3), (6), and (8) we obtain from Theorem A

$$N(n) > x - 1 \tag{9}$$

for arbitrary $x$ and sufficiently large $n$.

**3. Numerical estimates on the lower bound of $N(n)$.** In addition to Theorems A and B we need a result of Brun's sieve method. We shall use the following theorem due to H. Rademacher **(1)**.

THEOREM C. *Let $P(D; x; p_1, \ldots, p_r)$ denote the number of positive integers, $y$, no greater than $x$ which lie in an arithmetic progression $\Lambda + tD (t = 0, 1, \ldots,)$ where $0 < \Lambda < D$ and $(\Lambda, D) = 1$ and so that $y \not\equiv a_i \pmod{p_i}$, $y \not\equiv b_i \pmod{p_i}$ $(i = 1, \ldots, r)$.*

*If $p_1 < \ldots < p_r$ are primes with $p_i > 7$, then*

$$P(D; x; p_1, \ldots, p_r) > \frac{Cx}{D \log^2 p_r} - C' p_r^{79/10}$$

*where $C$ and $C'$ are positive constants.*

We shall also need the following simple fact.

LEMMA D. *The number of integers, $y$, no greater than $x$ which are divisible by a prime factor $p$ of $n$ so that $p > n^c$, is no greater than $x/cn^c$.*

*Proof.* Obviously there are at most $x/p$ numbers $y$ divisible by $p$ and therefore the number in question is no greater than

$$x \sum_{\substack{p \mid n \\ p > n^c}} \frac{1}{p} < x \sum_{p/n} \frac{1}{n^c} < x \frac{1}{n^c} \cdot \frac{1}{c} = x/cn^c$$

since there are less than $1/c$ prime factors of $u$ which exceed $n^c$.

*Case I. $n$ is even.* Pick $k$ so that

$$k \equiv -1 \pmod{2^{\left[\frac{1}{91} \log_2 n\right]}}, \; k \equiv 1 \pmod{15}; \tag{10}$$

$$k \not\equiv 0 \text{ or } -1 \pmod{p} \text{ for } p \text{ prime}, 7 < p < n^{1/90}; \; k < n^{1/10}.$$

We note that this restricts $k$ to an arithmetic progression with difference

$$D = 15.2^{\left[\frac{1}{91} \log_2 n\right]} < c_1 n^{1/91}.$$

Thus by Theorem C there are at least

$$\frac{C n^{1/10}}{c_1 n^{1/91} \log^2 n \, (1/90)^2} - C' n^{\frac{79}{10} \cdot \frac{1}{90}} \tag{11}$$

$$= c_2 n^{81/910}/\log^2 n - C' n^{79/900} > c_3 n^{81/910}/\log^2 n$$

choices of $k$.

According to Lemma D the number of natural numbers below $n^{1/10}$ which have a prime factor greater than $n^{1/90}$ in common with $n$ does not exceed $90\, n^{8/90}$. Since $81/910 > 8/90$, it follows from (11), and the fact that $k$ has no factors less than $n^{1/90}$, that we can choose $k$ so that

$$(k, n) = 1. \tag{12}$$

From (10) and Theorem B it follows that

$$N(k) > n^{1/90} - 1 > \tfrac{1}{3} n^{1/91}, \tag{13}$$

$$N(k + 1) > \min\{\tfrac{1}{2} n^{1/91}, n^{1/90}\} - 1 > \tfrac{1}{3} n^{1/91},$$

for $n$ sufficiently large.

We now set $n = n_1 + n_2 k$ where $0 < n_1 < k$ and let $u = n_1 + u_1 k$, where we pick $u_1$ subject to the following conditions.

$$u_1 \not\equiv n_1 \pmod{2}; \tag{14}$$

$$\left.\begin{array}{l} u_1 \not\equiv -n_1/k \pmod{p}, \; p \nmid k \\ u_1 \not\equiv n_2 \pmod{p} \end{array}\right\} \quad p \text{ prime}, 3 < p < k;$$

$$u_1 < n^{169/200}.$$

Note that the incongruence (mod 2) implies that $u$ is odd.

The incongruences modulo 2, 3, and 5 can be satisfied by restricting $u_1$ to a progression with difference 30. In order to apply Theorem C we need $(u_1, 30) = 1$. If $(u_1, 30) > 1$ we write $u_1 = u_1' \,.\, (u_1, 30)$ with $(u_1', 30) = 1$; then according to Theorem C, the number of such choices of $u_1'$ is at least

$$(15) \qquad \frac{C\, n^{159/200}}{30 \log^2 k} - C' k^{79/10} > c_4\, n^{159/200}/\log^2 n - C' n^{79/100} > 0$$

for $n$ sufficiently large.

From (14) we see that $u$ is not divisible by any prime less than $k$ and prime to $k$. If $u$ were divisible by a prime $p$ which divides $k$, then $n_1$, and hence $n$, would be divisible by $p$, in contradiction to (12). Hence from (13) we obtain

$$(16) \qquad N(u) \geqslant k > N(k) > \tfrac{1}{3}n^{1/91} \text{ for sufficiently large } n.$$

Also, if we set $m = (n - u)/k$, then

$$(17) \qquad \begin{aligned} m &> n/n^{1/10} - (1 + n^{159/200}) > \tfrac{1}{2}n^{9/10} \\ &> n^{1/10} + (1 + n^{159/200}) > u > 1 \end{aligned}$$

for sufficiently large $n$. Finally, according to (12) and (14), all prime factors of $m$ exceed $k$ so that

$$(18) \qquad N(m) \geqslant k > N(k) > \tfrac{1}{3}n^{1/91}.$$

According to (17) and (18) our choice of $k$, $u$, $m$ satisfies the conditions of Theorem A. Thus by (13), (16), and (18) we have

$$(19) \qquad N(n) > \tfrac{1}{3}n^{1/91} \text{ for all sufficiently large even } n.$$

*Case II. n odd.* Instead of applying Theorem C to $k$ we apply it to $k + 1$ with equation (10) replaced by

$$(10') \qquad \begin{aligned} &k + 1 \equiv 1 \ (\text{mod } 2^{[\frac{1}{91}\log_2 n]})\ k + 1 \equiv 2 \ (\text{mod } 15); \\ &k + 1 \not\equiv 0 \text{ or } 1 \ (\text{mod } p) \text{ for } 7 \leqslant p \leqslant n^{1/90}; \\ &k + 1 \leqslant n^{1/10}. \end{aligned}$$

The rest of the argument on $k$ proceeds as before and equation (12) remains unchanged, while (13) becomes

$$(13') \qquad \begin{aligned} N(k) &> \min\{\tfrac{1}{2}n^{1/91}, n^{1/90}\} - 1 > \tfrac{1}{3}n^{1/91}, \\ N(k + 1) &> n^{1/90} - 1 > \tfrac{1}{3}n^{1/91}. \end{aligned}$$

The choice of $u$ is modified so that (14) is replaced by

$$(14') \qquad u_1 \not\equiv n_2 \ (\text{mod } 2); \qquad u_1 \not\equiv -\, n_1/k \ (\text{mod } p)$$

for all primes $3 \leqslant p \leqslant k$ which do not divide $k$; while $u_1 \not\equiv n_2 \ (\text{mod } p)$ for all primes $3 \leqslant p \leqslant k$; $u_1 < n^{159/200}$.

It then follows from (13′) and (14′) that both $n$ and $m = (n - u)/k$ are odd, and the remainder of the argument proceeds exactly as before to yield the following.

THEOREM. *There exists a number $n_0$ so that for all $n > n_0$ we have*

$$N(n) > \tfrac{1}{3}n^{1/91}.$$

**4. Remarks.** The exponent 1/91 in our result is far from best possible. We have not used the best available sieve method, nor have we even squeezed the last drop out of the sieve method quoted. It seems, however, reasonable to defer such efforts in the hope that other theorems of the type of Theorem A can be developed, which may eliminate the twofold use of the double sieve of Theorem C. This would be accomplished, for example, if either the occurrences of both $N(k)$ and $N(k + 1)$ or the inequality $N(m) + 1 \geqslant k$ could be eliminated.

Theorem A can never lead to $N(n) \geqslant n^{1/2}$ since we must have $n > mk$ and $N(m) + 1 \geqslant k$ so that $k \leqslant m \leqslant n^{1/2}$ and $N(k) < n^{1/2}$.

On the other hand, our result seems to eliminate the possibility of a reasonable modification of MacNeish's conjecture which would express $N(n)$ in terms of prime power divisors of $n$; since for any positive $c$ there are infinitely many $n$ for which even the greatest prime power divisor is less than $n^c$.

REFERENCE

1. H. Rademacher, *Beiträge zur Viggo Brunschen Methode in der Zahlentheorie*, Abbh. Math. Sem. Hamburg, *3* (1924), 12–30.

*Number Theory Institute* (1959), *Boulder, Colorado*
*University of Colorado*
*University of California, Los Angeles*

# A SIMPLE CLOSED CURVE IS THE ONLY HOMOGENEOUS BOUNDED PLANE CONTINUUM THAT CONTAINS AN ARC

R. H. BING

**1. Introduction.** One of the unsolved problems of plane topology is the following:

*Question.* What are the homogeneous bounded plane continua?

A search for the answer has been punctuated by some erroneous results. For a history of the problem see **(6)**.

The following examples of bounded homogeneous plane continua are known: a point; a simple closed curve; a pseudo arc **(2, 12)**; and a circle of pseudo arcs **(6)**. Are there others?

The only one of the above examples that contains an arc is a simple closed curve. In this paper we show that there are no other such examples. We list some previous results that point in this direction. Mazurkiewicz showed **(11)** that the simple closed curve is the only non-degenerate homogeneous bounded plane continuum that is locally connected. Cohen showed **(8)** that the simple closed curve is the only homogeneous bounded plane continuum that contains a simple closed curve. Cohen showed **(8)** that the simple closed curve is the only non-degenerate homogeneous bounded plane continuum that is arcwise connected.

In this paper we prove the following theorem:

THEOREM 1. *The simple closed curve is the only homogeneous bounded plane continuum that contains an arc.*

Theorem 1 is proved by listing certain properties possessed by any homogeneous bounded plane continuum that contains an arc but is not a simple closed curve (these properties with their consequences are listed in §§ 2, 3, 5, 6, and 10) and then showing (Theorem 6) that no homogeneous bounded plane continuum could have one of these properties. The proof of Theorem 1 is completed in § 10.

In this paper, all sets are assumed to be metric. For the most part we will deal with planar sets but since some of the results apply to more general metric spaces, we do not suppose that sets discussed are planar unless this is stated. We recall some definitions, related results, and related questions.

A set is *homogeneous* if for each pair of its points $p$, $q$ there is a homeomorphism of the set onto itself that takes $p$ to $q$.

An $\epsilon$ collection is a collection each of whose elements is of diameter no more than $\epsilon$.

Received April 2, 1959.

An $\epsilon$ chain is a finite ordered $\epsilon$ collection of open sets $d_1, d_2, \ldots, d_n$ such that $d_i$ intersects $d_j$ if and only if $i$ is adjacent to $j$.

A compact continuum $X$ is snake-like if for each positive number $\epsilon$, $X$ can be covered by an $\epsilon$ chain. It is known **(5)** that the only non-degenerate homogeneous snake-like continuum is a pseudo arc.

It is convenient to associate with any open covering $G$ a 1-complex $C(G)$, called the 1-nerve of $G$, such that there is a $1-1$ correspondence between the elements of $G$ and the vertices of $C(G)$ and two elements of $G$ intersect if and only if the corresponding vertices of $C(G)$ are joined by a 1-simplex in $C(G)$. Note that the 1-nerve of an $\epsilon$ chain is topologically an arc.

A compact continuum $X$ is tree-like if for each positive number $\epsilon$ there is an $\epsilon$ collection $G$ of open sets covering $X$ such that the 1-nerve of $G$ contains no simple closed curve. Each 1-dimensional compact plane continuum that does not separate the plane is tree-like **(3)**.

*Question.* Is there a homogeneous tree-like continuum that contains an arc?

Jones has shown **(10)** that each homogeneous tree-like compact continuum is indecomposable. Perhaps each is hereditarily indecomposable.

A compact continuum $X$ is circle-like if it is not snake-like but for each positive number $\epsilon$ there is an $\epsilon$ collection $G$ of open sets irreducibly covering $X$ such that the 1-nerve of $G$ is topologically a circle. A simple closed curve and a circle of pseudo arcs are examples of circle-like homogeneous planar continua. Example 2 of **(4)** is not known to be non-homogeneous. A solenoid is an example of a compact homogeneous continuum that contains an arc. However, the simple closed curve is the only solenoid that is planar.

A solenoid may be defined as the intersection of a sequence of tori $T_1, T_2, \ldots,$ such that $T_{i+1}$ runs smoothly around inside $T_i$ $n_i$ times longitudinally without folding back and $T_i$ has cross diameter of less than $1/i$. The sequence $n_1, n_2, \ldots,$ determines the topology of the solenoid. If it is $1, 1, \ldots,$ after some place, the solenoid is a circle. If it is $2, 2, \ldots,$ the solenoid is the dyadic solenoid.

There is no loss of generality in supposing that each integer in the sequence $n_1, n_2, \ldots,$ used in defining a solenoid is prime, for if $n_i$ is not prime, it may be replaced in the sequence by its prime factors. The order of the elements of the sequence does not affect the topology of the solenoid—that is, if $m_1, m_2, \ldots,$ is a reordering of $n_1, n_2, \ldots,$ the solenoids determined by the sequences are topologically equivalent. Also, the first few terms of the sequence does not affect the topology of the solenoid. Hence, solenoids determined by the sequences of primes $n_1^1, n_2^1, \ldots,$ and $n_1^2, n_2^2, \ldots,$ are topologically equivalent if it is possible to remove a finite number of elements from each so that each prime greater than 1 occurs the same number of times in each of the remainders. Perhaps the converse of this is true.

Another way of describing a solenoid is to consider a unit circle $C$ in the plane with centre at the origin and a sequence of maps $f_1, f_2, \ldots,$ of $C$ onto itself so that in polar co-ordinates

$$f_i(1, \theta) = (1, n_i\theta).$$

This solenoid is the inverse limit of the circles and the $f_i$'s and consists of all points $p_1 \times p_2 \times p_3 \times \ldots$ of the Cartesian product $C \times C \times C \times \ldots$ such that for each $i$, $p_i = f_i(p_{i+1})$.

We show in Theorem 9 that if a circle-like homogeneous continuum contains an arc, it is a solenoid. Although a solenoid may not be planar, it is locally planar. Anderson has shown **(1)** that the only 1-dimensional locally connected continuum that is not locally planar at any point is the Menger universal curve. It is homogeneous **(1)**.

*Question.* Are solenoids and the Menger universal curve the only homogeneous 1-dimensional compact continua that contain arcs?*

**2. Some elementary properties of $M$.** In §§ 2 and 3 we suppose that $M$ is a homogeneous, non-degenerate, bounded plane continuum that contains an arc but is not a simple closed curve. Our plan is to list enough of the properties of such an assumed $M$ to show that it cannot exist. This section lists some elementary properties of $M$.

*Property* 1. *$M$ is not locally connected.* Mazurkiewicz **(11)** showed that the simple closed curve is the only non-degenerate homogeneous bounded locally connected plane continuum.

*Property* 2. *$M$ is not connected im kleinen at any point.* A set $X$ is connected *im kleinen* at a point $x$ if for each neighbourhood $U$ of $x$ there is a neighbourhood $N$ of $x$ such that $N \cdot X$ lies in the component of $U \cdot X$ containing $x$. If $M$ were connected *im kleinen* at one point, it would follow from the homogeneity of $X$ that it is connected *im kleinen* at every point. Since a continuum is locally connected at each point if it is connected *im kleinen* at each point, Property 1 implies Property 2.

*Property* 3. *$M$ contains an open set $U$ with uncountably many components.* Property 3 follows from Property 2 and the following theorem.

THEOREM 2. *If a complete metric space fails to be connected im kleinen at each point of a dense $G_\delta$ set, it contains an open set $U$ with uncountably many components.*

*Proof.* Suppose $X$ is a complete metric space that fails to be connected *im kleinen* at each point of a dense $G_\delta$ set $Y$ and $Y$ is the intersection of the open sets $U_1, U_2, \ldots$.

*At the 1959 Summer Meeting of the American Mathematical Society J. H. Case presented an abstract announcing another such continuum.

Assume that $X$ fails to contain an open set $U$ with uncountably many components. Let $V_1$ be an open subset of $U_1$ of diameter less than $\frac{1}{2}$. It follows from the Baire Category Theorem that $V_1$ contains an open subset $V_2$ such that

$$\bar{V}_2 \subset V_1 \cdot U_2,$$
$$\text{diameter } V_2 < 1/2^2, \text{ and}$$
$$V_2 \text{ lies in a component of } V_1.$$

Similarly, there is an open set $V_3$ satisfying

$$\bar{V}_3 \subset V_2 \cdot U_3,$$
$$\text{diameter } V_3 < 1/2^3,$$
$$V_3 \text{ lies in a component of } V_2.$$

If one continues to get $V_4, V_5, \ldots,$ one finds that $X$ is connected *im kleinen* at $\bar{V}_1 \cdot \bar{V}_2 \ldots$. This contradicts the fact that $\bar{V}_1 \cdot \bar{V}_2 \ldots$ is a point of $Y$.

Grace **(9)** has given an example of a compact metric continuum that fails to be locally connected anywhere but which contains no open subset with uncountably many components. This shows that Theorem 2 cannot be weakened by replacing the property of not being connected *im kleinen* with the property of not being locally connected.

*Property* 4. *M contains no simple triod.* A simple triod is the sum of three arcs such that the intersection of any two of them is the same point $p$. If $M$ contained a simple triod, it would follow from the homogeneity of $M$ that each component of $U$ of Property 3 would contain a simple triod. This would violate the fact that the plane does not contain uncountably many mutually exclusive simple triods. See Theorem 4.

*Property* 5. *M contains no simple closed curve.* Cohen showed **(8)** that if a bounded homogeneous plane continuum contains a simple closed curve, it is one.

## 3. Arc components of $M$.

An arc component of a set $X$ is a subset of $X$ maximal with respect to the property that each pair of points of the subset belongs to an arc in $X$. In this section we show that the closure of each arc component of the assumed homogeneous bounded plane continuum $M$ is homogeneous. In doing this we find it convenient to work with only certain parts of the arc components. These parts are called rays and are defined as follows.

Suppose $p$ and $q$ are two points of the same arc component of $M$. The sum of all arcs in $M$ that have $p$ as end point and contain $q$ is called a *ray* starting at $p$. One may note that this ray differs from an ordinary ray of the plane in that it is neither straight nor closed. However, it has a starting point and is the image of an ordinary ray under a $1-1$ continuous transformation.

*Property* 6. *Each ray in $M$ is the sum of a countable number of arcs.* Let $p$ be the starting point of a ray $R$ and $\{p_i\}$ be a countable dense subset of $R$. Then $R$ is the sum of the arcs $pp_1, pp_2 \dots$. If there were a point $r$ of $R$ not in any $pp_i$, we would consider the arc $pr$. It follows from the homogeneity of $M$ that $r$ is the interior point of an arc so $pr$ may be extended to an arc $ps$ so that $r$ is contained in the interior of $ps$. Since $M$ contains no simple triod, each $p_i$ belongs to the arc $pr$. However, $\{p_i\}$ would not be dense in $R$ since no $p_i$ is near $s$.

*Property* 7. *For each point $p$ of an arc component $A$ of $M$, $A$ is the sum of two rays $R_1$, $R_2$ starting at $p$ such that $R_1 \cdot R_2 = p$.* It follows from the homogeneity of $M$ that $p$ is an interior point of an arc $ab$. It follows from the fact that $M$ contains no simple triod (Property 4) that $A$ is the sum of two rays starting at $p$ and going through $a$, $b$ respectively. Since $M$ contains no simple closed curve (Property 5), these rays intersect only at $p$.

*Property* 8. *$M$ has uncountably many arc components.* If $M$ had only countably many arc components, it would follow from Properties 6 and 7 that $M$ is the sum of a countable collection of arcs. It would then follow from the Baire Category Theorem that one of these arcs contains an open subset of $M$. The homogeneity of $M$ would then imply that $M$ is a 1-manifold. However, the simple closed curve is the only compact connected 1-manifold.

*Property* 9. *If $R$ is a ray of $M$ and $p$ is a point of $\bar{R}$, one of the rays starting at $p$ lies in $\bar{R}$.* If neither of the rays starting at $p$ lies in $\bar{R}$, $p$ belongs to an arc $ab$ in $M$ such that neither $a$ nor $b$ is a point of $\bar{R}$. It follows from Property 8 and the homogeneity of $M$ that there is an uncountable family $\{h_\alpha\}$ of homeomorphisms of $(ab + \bar{R})$ into $M$ such that if $\alpha \neq \beta$, $h_\alpha(ab)$, $h_\beta(ab)$ belong to different arc components of $M$.

It follows from Theorem 3 of § 4 that there is an arc $A_0$ of the collection $\{h_\alpha(ab)\}$ with two sequences $A_1, A_3, \dots$, and $A_2, A_4, \dots$, of arcs of $\{h_\alpha(ab)\}$ converging homeomorphically to $A_0$ from opposite sides. For convenience we suppose $A_0 = ab$. Some two of the arcs $A_{2i}$, $A_{2i+1}$ near $A_0$ would separate some point of $\bar{R}$ from $A_0$ in $\bar{R}$ and hence two points $p$, $q$ of $R$ from each other in $R$. But then the arc $pq$ in $R$ would cross either $A_{2i}$ or $A_{2i+1}$ and violate Property 4.

*Property* 10. *If $\bar{R}_1$ is the closure of a ray of $M$, it contains a continuum $\bar{R}$ that is irreducible with respect to being the closure of a ray.* Let $D_1, D_2, \dots$, be a countable basis of open sets for the plane and $R_1, R_2, \dots$, be a sequence of rays such that

1. $R_{i+1}$ is a ray in $\bar{R}_i$ missing $D_i$ if any ray in $\bar{R}_i$ misses $D_i$; $R_{i+1} = R_i$ if each ray in $\bar{R}_i$ intersects $D_i$.

If $p$ is a point common to the elements of the decreasing sequence $\bar{R}_1, \bar{R}_2, \bar{R}_3, \dots$, it follows from Property 9 that one of the rays $R$ starting at $p$ lies in infinitely many of the $\bar{R}_i$'s. Hence it lies in $\bar{R}_1 \cdot \bar{R}_2 \cdot \bar{R}_3 \dots$. If $R'$ is a ray

in $\bar{R}$, it follows from Condition 1 above that $R'$ intersects each $D_i$ that $R$ intersects. Hence $\bar{R}' = \bar{R}$.

*Property* 11. *If $R$ is a ray in an arc component $A$ of $M$, $\bar{R} = \bar{A}$.* Assume $p$ is a point in $A - \bar{R}$. It follows from Property 10 and the homogeneity of $M$ that $p$ is the starting point of a ray $R'$ whose closure is irreducible with respect to being the closure of a ray. Then $\bar{R}'$ does not contain $R$ and there is a point $q$ of $R - \bar{R}'$. Let $R''$ be a ray starting at $q$ whose closure is irreducible with respect to being the closure of a ray. Since neither of the rays $R'$, $R''$ contains the other and the starting point $q$ of $R''$ does not belong to $R'$, the rays $R'$, $R''$ do not intersect. Either some point of $pq$ belongs to both $\bar{R}'$ and $\bar{R}''$ or some point of $pq$ belongs to neither. We show that in either case, the assumption that $\bar{R} \neq \bar{A}$ has led to a contradiction.

If some point $r$ of the arc $pq$ fails to belong to $\bar{R}' + \bar{R}''$, there is no ray starting at $r$ whose closure is irreducible with respect to being the closure of a ray. This violates Property 10 and the homogeneity of $M$.

If some point $r$ of $pq$ belongs to both $\bar{R}'$ and $\bar{R}''$, there are two mutually exclusive rays in $A$ each missing $r$ and such that $r$ belongs to the closure of each. This violates the homogeneity of $M$ since there are not two mutually exclusive rays in $A$ each missing $q$ such that $q$ belongs to the closure of each.

*Property* 12. *If the closures of two arc components of $M$ intersect, the closures are equal.* Suppose $A_1, A_2$ are two arc components whose closures contain the point $p$. Let $A_p$ be the arc component of $M$ containing $p$. It follows from Property 9 that one of the rays starting at $p$ lies in $\bar{A}_1$ and from Property 11 that $A_p$ lies in $\bar{A}_1$. Similarly $A_p$ lies in $\bar{A}_2$. It follows from Property 10, Property 11, and the homogeneity of $M$ that the closure of each arc component of $M$ is irreducible with respect to being the closure of an arc component. Hence, $\bar{A}_p = \bar{A}_1 = \bar{A}_2$.

*Property* 13. *The closure of each arc component $A$ of $M$ is homogeneous.* We show that $\bar{A}$ is homogeneous by showing that if $p$ is a point of $A$ and $q$ is a point of $\bar{A}$, there is a homeomorphism of $\bar{A}$ onto itself taking $p$ to $q$. The homogeneity of $M$ implies that there is a homeomorphism $h$ of $M$ onto itself taking $p$ to $q$. Since such a homeomorphism takes arc components onto arc components, it follows from Property 12 that $h(\bar{A}) = \bar{A}$.

If $q$ and $r$ are points of $\bar{A} - A$ and one wishes a homeomorphism of $\bar{A}$ onto itself taking $q$ onto $r$, one could use the preceding paragraph to show that there is a homeomorphism $h_1$ of $\bar{A}$ onto itself taking $q$ to $p$ and a homeomorphism $h_2$ of $\bar{A}$ onto itself taking $p$ to $r$. The required homeomorphism is $h_2h_1$.

**4. Collections of arcs in the plane.** In this section we digress from our consideration of homogeneity to consider collections of arcs in the plane.

Theorem 3 is used in establishing Properties 9 and 15 but is of interest aside from these applications.

We recall the following notions concerning the abutting of arcs in the plane $E^2$ Suppose $ab$, $cd$, and $ef$ are arcs in $E^2$ such that $ab \cdot cd = c$ and $ab \cdot ef = e$ are interior points of $ab$. Then $cd$ and $ef$ are said to ***abut on opposite sides*** of $ab$ if there is a homeomorphism of $E^2$ onto itself that takes $ab$ onto a horizontal segment and $cd$, $ef$ onto vertical segments which lie except for their points of contact with $ab$ on opposite sides of the line containing $ab$.

A sequence of arcs $A_1, A_2, \ldots,$ is said to ***converge homeomorphically*** to an arc $A_\infty$ if for each positive number $\epsilon$ there is an integer $n$ such that if $n < i$, there is a homeomorphism of $A_i$ onto $A_\infty$ that moves no point more than $\epsilon$.

Suppose $ab$, $cd$, $ef$ are arcs such that $cd$ and $ef$ abut on $ab$ from opposite sides. A sequence of arcs $A_1, A_2, \ldots,$ converging homeomorphically to $ab$ is said to ***converge homeomorphically from the cd side of ab*** if none of the arcs intersect $ab$ and all but possibly a finite number of these arcs intersect $cd$. Two sequences or arcs converging homeomorphically to $ab$ are said to ***converge homeomorphically from opposite sides*** if one of the sequences converges from the $cd$ side of $ab$ and the other from the $ef$ side of $ab$.

THEOREM 3. *If $W$ is an uncountable collection of mutually exclusive arcs in $E^2$, then there is an element $w$ of $W$ and two sequences of elements of $W$ converging homeomorphically to $w$ from opposite sides.*

This result follows as a corollary of the following result which has a more cumbersome statement.

THEOREM 3'. *Each uncountable collection of mutually exclusive arcs in $E^2$ has a countable subcollection $W'$ such that each element $w_0$ of $W - W'$ has the following property:*

*For each pair of arcs $cd$, $ef$ abutting on $w_0$ from opposite sides and each positive number $\epsilon$ there are uncountable subcollections $W_1$, $W_2$ of $W - W'$ such that*

1. *each element of $W_1$ intersects $cd$,*
2. *each element of $W_2$ intersects $ef$, and*
3. *for each element $w$ of $W_1 + W_2$ there is a homeomorphism of $w$ onto $w_0$ that moves no point by more than $\epsilon$.*

*Proof of Theorem* 3'. Let $W'$ be the collection of all elements $w$ of $W$ with the property that there is an arc $cd$ abutting on $w$ from one side and a positive number $\epsilon$ such that no uncountable subcollection $W_1$ satisfies Conditions 1 and 3 of the statement of Theorem 3'. Theorem 3' is established by showing that the collection W' does not have uncountably many elements. Assume $W'$ is uncountable.

For each element $w_a$ of $W'$ let $v_a$ be an arc abutting on $w_a$ from one side and $\epsilon_a$ be a positive number such that

4. $v_a$ intersects only a countable number of elements $w$ of $W'$ such that there is a homeomorphism of $w_a$ onto $w$ that moves no point by more than $\epsilon_a$.

Let $\epsilon'$ be a positive number so small that for an uncountable subcollection $W''$ of $W'$, $\epsilon'$ will serve as the $\epsilon_\alpha$ for each element $w_\alpha$ of $W''$.

Suppose $T$ is a triod which is the sum of an arc $ab$ and an arc $cd$ abutting on $ab$ from one side. For each element $w_\alpha$ of $W''$ let $h_\alpha$ be a homeomorphism of $ab + cd = T$ onto $w_\alpha + v_\alpha$ that takes $ab$ onto $w_\alpha$. Let $\rho$ denote the ordinary distance function for the plane. The homeomorphisms $h_\alpha$ may be regarded as points of a function space metrized as follows:

$$D(h_\alpha, h_\beta) = \max_{t \in T} \rho(h_\alpha(t), h_\beta(t)).$$

Then $\{h_\alpha\}$ is a separable metric space and some element $h_0$ of it is a limit point of an uncountable order (each neighbourhood of $h_0$ contains uncountably many points of $\{h_\alpha\}$).

Let $H$ be the set of all elements of $\{h_\alpha\}$ within $\frac{1}{2}\epsilon'$ of $h_0$ and $W'''$ be the set of all elements of $W''$ that are images of $ab$ under an element of $H$. We note that if $w_1$, $w_2$ are two elements of $W'''$ then there is a homeomorphism of $w_1$ onto $w_2$ that moves no point by more than $\epsilon'$.

For convenience we suppose that $h_0(ab) = w_0$ is the horizontal diameter of a unit circle $C$ with centre at the origin and $h_0(cd) = v_0$ is a vertical radius of $C$ which extends upward. Also, we suppose $\epsilon' < 1$.

Since each element of $H$ is within $\frac{1}{2}\epsilon'$ of $h_0$, each element of $W'''$ intersects the $y$-axis. Let $p_\alpha$ be the highest point where $w_\alpha$ intersects this axis. Let $p_\gamma$ be one of these $p_\alpha$'s which has uncountably many other $p_\alpha$'s above it. But then $v_\gamma$ intersects all of the $w_\alpha$'s such that $p_\alpha$ lies above $p_\gamma$. This contradicts the definition of $v_\gamma$ given in Condition 4. The assumption that $W'$ was uncountable led to this contradiction.

THEOREM 4. *Suppose $B, B_1, B_2, \ldots,$ is a sequence of mutually exclusive arcs in $E^2$ such that $B_1, B_3, \ldots,$ and $B_2, B_4, \ldots,$ converge homeomorphically to $B$ from opposite sides. If $C$ is a continuum intersecting each $B_i$ but neither end of $B$ and $h$ is a homeomorphism of $C + B + B_1 + B_2 + \ldots$ into $E^2$, then $h(B_1), h(B_3), \ldots,$ and $h(B_2), h(B_4), \ldots,$ converges homeomorphically to $h(B)$ from opposite sides.*

*Proof.* The proof is divided into two steps.

*Step* 1. *$C$ contains two subcontinua $C_1$, $C_2$ such that $C_1$ intersects all but possibly a finite number of the odd $B_i$'s but no even $B_i$ and $C_2$ intersects all but possibly a finite number of the even $B_i$'s but no odd $B_i$.* With no loss of generality we suppose that $B$ is the horizontal interval $ab$, that each odd $B_i$ intersects the perpendicular bisector of $ab$ at a point above $ab$ and each even $B_i$ intersects this perpendicular bisector at a point below $C$. The two continua $C_1$, $C_2$ that we describe will lie except for their intersections with $ab$ on opposite sides of the line containing $ab$.

Let $\epsilon$ be a positive number so small that neither $a$ nor $b$ lies within $\epsilon$ of $C$. Let $K_1$, $K_2$ be circles with centres at $a$, $b$ respectively with radii equal to

$\frac{1}{2}\epsilon$. Since we can throw away a finite number of $B_i$'s, we suppose that for each $B_i$ there is a homeomorphism of $B_i$ onto $B$ that moves no point by more than $\frac{1}{2}\epsilon$.

Let $X_i$ be an arc of $B_i$ irreducible from $K_1$ and $K_2$ and $Y_i$ be the arc from $a$ to $b$ obtained by adding to $X_i$ a radius of $K_1$ and a radius of $K_2$. Each of $Y_1, Y_2, \ldots,$ lies except for $a$, $b$ above $ab$. We suppose that the ordering is such that $Y_{2i+1}$ is above (except at $a$, $b$) $Y_{2i+3}$.

Let $D$ be the disc bounded by $ab + y_1$. If each arc in $D$ from $a$ to $b$ intersects $C$, it follows from the unicoherence of $D$ that some component $C_1$ of $D \cdot C$ separates $a$ from $b$ in $D$. This continuum $C_1$ intersects each $Y_{2i+1}$ and is the $C_1$ promised in Step 1. If there is an arc in $D$ from $a$ to $b$ that misses $C$, there is such an arc $Z$ which intersects $ab$ only at $a$, $b$. Let $D'$ be the disc bounded by $ab + Z$ and $X_{2i+1}$ be an arc on the interior of $D'$. Any component $C_1$ of $D' \cdot C$ that intersects $X_{2i+1}$ intersects each $X_{2j+1}$ $(j > i)$ and this $C_1$ will serve as the $C_1$ promised by Step 1. The continuum $C_2$ is obtained in a similar fashion.

*Step 2. If cd and ef are arcs abutting on $h(B)$ from opposite sides and infinitely many of the $h(B_{2i+1})$'s intersect cd, all but a finite number of the $h(B_{2i})$'s intersect ef.* We suppose with no loss of generality that $h(B) = ab$ is a horizontal segment, $cd$ is a vertical segment pointing upward from $ab$, and $ef$ is a vertical segment pointing downward from $ab$.

Let $\epsilon$ be a positive number so small that neither $a$ nor $b$ is within $\epsilon$ of $cd + ef + h(C_1) + h(C_2)$. Following Step 1, we let $K_1$, $K_2$ be circles with centres at $a$, $b$ respectively and radii equal to $\frac{1}{2}\epsilon$. Since we can disregard any finite collection of the $h(B_i)$'s, we suppose with no loss of generality that there is a homeomorphism of each $h(B_i)$ onto $h(B)$ that moves no point by more than $\frac{1}{2}\epsilon$.

We let $X_i$ be a subarc of $h(B_i)$ irreducible from $K_1$ to $K_2$ and $Y_i$ be the arc from $a$ to $b$ obtained by adding to $X_i$ radii of $K_1$ and $K_2$ respectively. Since infinitely many of the $h(B_{2i+1})$'s intersect $cd$, infinitely many of the $Y_{2i+1}$'s lie above $ab$ (except for $a$, $b$).

Suppose $Y_1$ lies above $ab$ (except for $a$, $b$) and let $D$ be the disc bounded by $Y_1 + ab$. Since each point of Int $D$ is separated from $ab$ by a $Y_{2i+1}$ and each $Y_{2i+1}$ misses $C_2$, $C_2$ does not intersect the interior of $D$. Since no $X_{2i}$ lies interior to $D$ (each intersects $C_2$), all but a finite number of these $X_{2i}$'s lie below $ab$. Hence, all but a finite number of the $X_{2i}$'s (and hence the $h(B_{2i})$'s) intersect $ef$.

Since all but a finite number of the $h(B_i)$'s intersect $cd + ef$ we suppose with no loss of generality that infinitely many $h(B_{2i+1})$'s intersect $cd$. It follows from Step 2 that all but a finite number of the $h(B_{2i})$'s intersect $ef$ and by a repetition of Step 2 that all but a finite number of the $h(B_{2i+1})$'s intersect $cd$.

It is known that the plane does not contain uncountably many mutually

exclusive triods (**3**, Theorem 5, p. 254). In extending this result to higher dimensions it is convenient to think of a simple triod as having a topological 1-simplex as base and having a feeler sticking out from an interior point of this base. The following theorem is a strengthening of this result concerning triods in the plane.

THEOREM 5. *Suppose $W$ is an uncountable collection of simple triods in the plane such that each of these triods has a designated base and feeler. If no two of the bases of the elements of $W$ intersect, some feeler intersects uncountably many bases.*

*Proof.* If the bases are mutually exclusive, it follows from Theorem 3′ that there is a base $b_0$ with uncountably many bases arbitrarily close on either side of $b_0$. The feeler from $b_0$ would intersect uncountably many of these nearby bases.

**5. The reduced continuum** $M'$. In this section we return to a study of the assumed homogeneous bounded plane continuum $M$ studied in §§ 2 and 3 which contains an arc but is not a simple closed curve. It follows from Properties 5 and 13 that if there is such an $M$, there is a continuum $M'$ that is the closure of one of its arc components. We list some properties that such an $M'$ would need to possess in order to show that there is no such $M'$ and hence no $M$. In §§ 5, 6, 10, we use $M'$ to denote a homogeneous bounded plane continuum one of whose arc components is dense in $M'$ but which is not a simple closed curve.

*Property* 14. *If $C$ is a non-degenerate subcontinuum of $M'$ that is not an arc, $C$ intersects uncountably many arc components of $M'$.* This is true by Property 8 if $C = M'$ so we suppose $C$ is a proper subcontinuum of $M'$. Let $p$ be a point of $M' - C$ and $A$ be the arc component of $M'$ containing $p$. Since each ray is dense in $M'$, there is a sequence of points $p_1, p_{-1}, p_2, p_{-2}, \ldots$, of $A - C$ such that $A$ is the sum of the arcs $p_ip_{i+1}$ and no two of the $p_ip_{i+1}$'s intersect except possibly at an end point of each. If one considers the intersections of these arcs $p_ip_{i+1}$ with $C$, one finds that $A \cdot C$ is the sum of a countable collection of mutually exclusive closed sets. Since no continuum is the sum of a countable number more than one of mutually exclusive closed point sets, $C$ intersects uncountably many arc components of $M'$.

*Property* 15. *Each non-degenerate proper subcontinuum* of *$M'$ is an arc.* If $M'$ contains a non-degenerate proper subcontinuum $C$ that is not an arc, it follows from Property 14 and the fact that each ray is dense in $M'$ that $M'$ contains an uncountable collection of mutually exclusive arcs each of which intersects $C$ but no one of which has an end on $C$. It follows from Theorem 3 of § 4 that there is one of these arcs $B$ that has two sequences of arcs $B_1, B_3, \ldots$, and $B_2, B_4, \ldots$, of the arcs converging homeomorphically to $B$ from opposite sides. It follows from Theorem 4 of § 4 that under no homeomorphism $h$ of

$C + B + B_1 + B_2 \ldots$ into the plane is the image of any interior point of $B$ accessible from the complement of $h(C + B + B_1 + B_2 + \ldots)$. This violates the homogeneity of $M'$ since some points of it are accessible from the complement of $M'$.

*Property* 16. *$M'$ is indecomposable.* If $M'$ were the sum of two proper subcontinua, it would follow from Property 15 that these subcontinua were arcs. The only homogeneous continuum that is the sum of two arcs is a simple closed curve.

**6. Continua each of whose proper subcontinua is an arc.** A solenoid is a non-degenerate homogeneous compact continuum each of whose proper subcontinua is an arc. Other examples are not at hand. We note that Property 15 shows that $M'$ has this property. The following question is related to the last two given in § 1.

*Question.* Are solenoids the only non-degenerate homogeneous compact continua each of whose proper subcontinua is an arc?

Theorems 7 and 9 answer this question in the affirmative for the cases of tree-like and circle-like continua.

In developing the following property, we use merely the fact that each proper subcontinuum of $M'$ is an arc (Property 15) rather than the facts that $M'$ is homogeneous and lies in the plane.

*Property* 17. *For each positive number $\epsilon$ and each arc $xy$ in $M'$ there is an $\epsilon$-chain $d_1, d_2, \ldots, d_n$ covering $xy$ such that $x$, $y$ belong to $d_1$, $d_n$ respectively and $M' \cdot Bd \sum d_i \subset \bar{d}_1 + \bar{d}_n$.* Let $e_1, e_2, \ldots, e_n$ be an $\epsilon$-chain covering $xy$ such that $x$, $y$ belong to $e_1$, $e_n$ respectively. There are open sets $O_1$, $O_2$ in $e_1$, $e_2$ respectively such that $xy$ is an arc component of $M' - (O_1 + O_2)$. It follows from Property 15 that $xy$ is a component of $M' - (O_1 + O_2)$.

Since no component of $M' - (O_1 + O_2)$ intersects both $xy$ and $M' - \sum e_i$, then $M' - (O_1 + O_2)$ is contained in two mutually exclusive open sets $A$, $B$ such that $xy \subset A$, $M' - \sum e_i \subset B$ (see **13**, Theorem 35, p. 21). The link $d_i$ of the chain $d_1, d_2, \ldots, d_n$ is defined to be $e_i \cdot (A + O_1 + O_2)$.

Since

$$M' \cdot \sum \bar{d}_i = M' \cdot A + M' \cdot (B + O_1 + O_2) \cdot \sum \bar{d}_i \subset \sum d_i + (\bar{O}_1 + \bar{O}_2),$$

one finds on subtracting $\sum d_i$ fromthe ends of the above inequality that

$$M' \cdot Bd \sum d_i \subset \bar{O}_1 + \bar{O}_2 \subset \bar{d}_1 + \bar{d}_n.$$

*Property* 18. *For each positive number $\epsilon$ there is a positive number $\delta$ such that if $ab$ is an arc in $M'$ such that $\rho(a, b) < \delta$, then either diameter $ab < \epsilon$ or $ab$ is $\epsilon$ dense in $M'$.* Assume that there is no such $\delta$. Then for each integer $i$ there is an arc $a_i b_i$ in $M'$ such that

$$\rho(a_i, b_i) < 1/i,$$
$$\text{diameter } a_i b_i > \epsilon,$$
$$a_i b_i \text{ is not } \epsilon \text{ dense in } M'.$$

Some subsequence of $a_1b_1, a_2b_2, \ldots,$ converges to a non-degenerate proper subcontinuum. Hence it is an arc. Some subarc of this arc is the limit of a folded sequence of arcs in $M'$ (each in an $a_ib_i$). The assumption that there is no $\delta$ leads to the contradiction of Theorem 6 of the next section.

*Property* 19. *If a point $p$ of $M'$ is accessible from a component $U$ of $E^2 - M'$, each point of any arc in $M'$ containing $p$ is accessible from the same side that $p$ is.* Let $xy$ be an arc containing the two points $p$, $q$ on its interior. With no loss of generality we suppose that $xy$ is horizontal, $q$ is between $p$ and $y$, and $rp$ is a vertical interval lying except for $p$ below $xy$ and in $U$. We show that $q$ is accessible from $U$ from below.

Let

$$\epsilon = \min \rho(r, p), \qquad \rho(x, p), \qquad \tfrac{1}{2}\rho(q, y),$$

$\delta$ be the positive number promised by Property 18, with $\delta < \epsilon$, and $D$ be a $\delta$-chain covering $xy$ and satisfying the conditions of Property 17. We use $D^*$ to denote the sum of the links of $D$. If $q$ is not accessible from $U$ from below, there is a point $s$ in $M' \cdot D^*$ which is beneath the point $q$. Let $ab$ be an arc in $M' \cdot D^*$ containing $s$ such that $ab$ lies below $xy$ and each end of $ab$ is in an end link of $D$. The arc $rp$ prevents either $a$ or $b$ from being in the end link of $D$ containing $x$ so $\rho(a, b) < \delta$. Also, $ab$ is not $\epsilon$ dense in $M'$ since it is not near $x$. However, diameter $ab > \epsilon$ since $\frac{1}{2}\rho(q, y) \geqslant \epsilon$. The assumption that $q$ was not accessible from $U$ from below led to a contradiction of Property 18.

**7. Folded sequences of arcs.** A solenoid is an example of a homogeneous continuum each of whose proper subcontinua is an arc. The arcs in this continuum seem to run in a parallel fashion and not to "zig-zag" or "fold back." We find from Theorem 6 that such folding is impossible in a homogeneous continuum each of whose proper subcontinua is an arc. No use is made of the fact that the continuum lies in the plane.

Suppose $a_1b_1, a_2b_2, \ldots,$ is a sequence of arcs converging (not necessarily homeomorphically) to an arc $xy$. The sequence is called a *folded sequence* converging to $xy$ if $a_1, b_1, a_2, b_2, \ldots,$ converges to $x$.

THEOREM 6. *Suppose $X$ is a homogeneous compact continuum each of whose proper subcontinua is an arc. Then no folded sequence of arcs in $X$ converges to an arc.*

*Proof.* Assume $a_1b_1, a_2b_2, \ldots,$ is a sequence of arcs converging to an arc $xy$ such that $a_1, b_1, a_2, b_2, \ldots,$ converges to $x$. If $\epsilon < \frac{1}{2}\rho(x, y)$, $y$ has the following property:

*Property* $(y, \epsilon)$. For each positive integer $n$ there is a $1/n$ chain $E$ such that $X$ intersects each link of $E$, the distance between $y$ and the first link of $D$ is less than $1/n$, the distance between the end links of $E$ is more than $\epsilon$, and $X \cdot Bd\, E^*$ lies in the closure of the last link of $E$.

We can obtain such an $E$ as follows.

1. Let $D$ be a $1/n$ chain covering $xy$ such that the first link of $D$ contains $y$, the last link contains $x$, and for each other link $d$, $\bar{d} \cdot X \subset D^*$. The existence of such a $D$ is guaranteed by the proof given in Property 17 of the last section.

2. Let $a_i b_i$ be an arc of the folded sequence converging to $xy$ such that each of $a_i$, $b_i$ lies in the last link of $D$, some point of $a_i b_i$ lies in the first link of $D$, and $D$ covers $a_i b_i$.

3. Let $D'$ be a chain covering $a_i b_i$ such that $D'$ refines $D$, the first and last link of $D'$ lie in the last link of $D$, some link of $D'$ lies in the first link of $D$, and $X \cdot Bd\, D'^*$ lies in the sum of the closures of the two end links of $D'$.

Then $E$ can be formed as follows. The $j$th link of $E$ is the sum of the elements of $D'$ in the $j$th link of $D$.

Let $X(\epsilon)$ be the set of all points $y$ of $X$ such that $y$ has Property $(y, \epsilon)$. Then $X(\epsilon)$ is closed and it follows from the homogeneity of $X$ that each point of $X$ belongs to some $X(\epsilon)$ for some $\epsilon$. It follows from the Baire Category Theorem that there is an integer $m$ and an open set $U$ such that $U \subset X(1/m)$.

We obtain a sequence of chains $E_1, E_2, \ldots$, such that

1. $E_i$ is a $1/i$ chain such that the distance between its end links is more than $1/m$,

2. $X \cdot Bd\, E_i^*$ lies in the closure of the last link of $E_i$, and

3. the first link $e_1^1$ of $E_1$ lies in $U$, and the closure of the first link $e_1^{i+1}$ of each $E_{i+1}$ lies in the first link of $E_i$.

The intersection of the $e_1^i$'s is a point that cannot be the interior point of any arc in $X$. This contradiction results from the false assumption that there is a folded sequence of arcs in $X$ converging to an arc.

The following result gives an immediate application of Theorem 6. The result is not needed in the proof of Theorem 1 but it can be used in lieu of Property 19 in finishing the proof of Theorem 1 for the case where $M$ does not separate the plane, since each 1-dimensional bounded plane continuum that does not separate the plane is tree-like **(3)**.

THEOREM 7. *There exists no non-degenerate, homogeneous, tree-like continuum each of whose proper subcontinua is an arc.*

*Proof.* Assume $X$ is a non-degenerate, homogeneous, tree-like continuum each of whose proper subcontinua is an arc. It follows from **(10)** that $X$ is indecomposable. Let $D_i$ be a $1/i$ tree-chain covering $X$ and $a_i b_i$ be an arc in $X$ such that both ends of $a_i b_i$ lie in the same link of $D_i$ and

$$\text{diameter } X/4 - 1/i < \text{diameter } a_i b_i < \text{diameter } X/2.$$

Such an arc $a_ib_i$ may be found by considering an arc of large diameter in $X$ both of whose ends lie in the same link of $D_i$, reducing this arc by throwing away the part of it in this link and considering one of the larger components of the remainder, reducing the component in a similar fashion, . . . , and stopping this reduction when an arc of the required diameter is found.

Some subsequence of $a_1b_1, a_2b_2, \ldots$, converges to a proper subcontinuum of $X$. It follows from the hypothesis that this subcontinuum is an arc $ab$. However, there is a folded sequence of arcs (each in one of the $a_ib_i$'s) converging to a subarc of $ab$. This contradicts Theorem 6.

**8. A nearly homogeneous example.** Consider the example $Y$ shown in Figure 1. At a glance it might appear to be homogeneous. The example $Y$ intersects the $x$ axis in a Cantor set and is the sum of semicircles with ends on this Cantor set. Also, the example may be obtained by starting with a punctured disc with three holes and digging canals into the disc from the four complementary domains of the punctured disc.

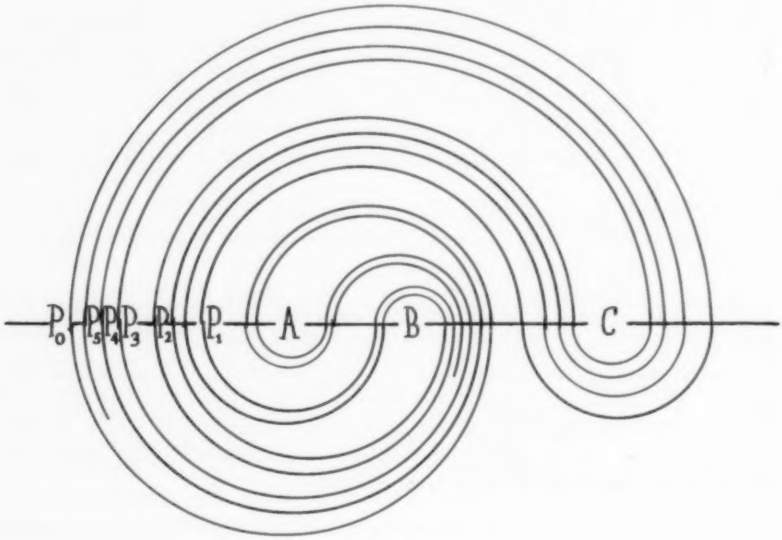


FIGURE 1

The canal from the unbounded complementary domain may be defined in terms of its right bank as follows. Let $p_0$ be the point furthest to the left of $Y$ and consider the ray $R$ starting at $p_0$, going along the upper semicircle of $X$, then along the lower right semicircle, and then down the right bank of the canal leading from the unbounded complementary domain of $Y$. Let $p_1$ be

the first point on $R$ which is between $p_0$ and $A$ on the $x$ axis, $p_2$ be the first point of $R$ that is between $p_0$ and $p_1$ on the $x$ axis, and in general, $p_{i+1}$ is the first point of $R$ that is between $p_0$ and $p_i$ on the $x$ axis.

As viewed from $C$, $p_0p_1$ circles $C$. However, it circles neither $A$ nor $B$ when viewed from $A$, $B$ respectively. Furthermore, $p_0p_2$ circles $B$ and $C$ but not $A$, $p_0p_3$ circles $B$, $p_0p_4$ circles $A$ and $B$, $p_0p_5$ circles $A$, $p_0p_6$ circles $A$ and $C$, $p_0p_7$ circles $C$.... The other canals from $A$, $B$, $C$ run between the canals from the unbounded complementary domains and each canal is dense in $Y$. Figure 1 does not show the banks of the canals from $A$, $B$, $C$ but shows only an arc on the outer bank. This arc does not separate the plane. There are points of $X$ not shown in the figure that are nearer $A$, $B$, $C$ than any point on the outer bank. These points keep the complementary domains containing $A$, $B$, $C$ from running into each other.

We could write the equation of $Y$ by giving functions $f, g$ such that $(x, f(x))$ are abscissas of the ends of semicircles of $Y$ in the upper half-plane and $(x, g(x))$ are the abscissas of the ends of semicircles of $Y$ in the lower half-plane. However, we shall not do this since we are interested in $Y$'s topological properties rather than its equation.

Example $Y$ is locally homogeneous—that is, for each pair of points $p$, $q$ of $Y$ there are arbitrarily small homeomorphic open subsets $N_p$, $N_q$ containing $p$, $q$ respectively. In fact, the open subsets may be taken to be homeomorphic with the Cartesian product of a Cantor set with an open interval.

Also, example $Y$ is nearly homogeneous—if $p$ and $q$ are points of $Y$, then for each open subset $U$ of $Y$ containing $q$ there is a homeomorphism of $Y$ onto itself taking $p$ into $U$. One may see that this is true since each arc component is dense in $Y$ and each arc lies in an open subset homeomorphic with the Cartesian product of a Cantor set and an open interval. See **(7)** for a discussion of various types of homogeneity.

However, $M$ is not homogeneous. If it were, for each positive number $\epsilon$ and each point $p$ there would be a homeomorphism $h$ of $Y$ onto itself such that $h$ moves no point by more than $\epsilon$ and $p$, $h(p)$ belong to different composants of $Y$. (See Theorem 8 of § 9.) Suppose that $\epsilon$ is taken to be less than the distance across the canal leading from the unbounded complementary domain at a wide point and $p$ is taken to be the highest point of $Y$. There is a canal leading from the outside that locally separates $p$ from $h(p)$ in the plane. As $p$ is moved parallel to this canal and in the direction of its wide spot, the canal continues to separate the moving $p$ from the corresponding $h(p)$. However, as the canal widens, it is no longer possible for $p$ to be within $\epsilon$ of its image under $h$.

This intuitive reason of why $Y$ is not homogeneous is refined in § 10 to establish Property 20 and finish the proof of Theorem 1.

**9. Homeomorphisms near the identity.** In indicating why the nearly homogeneous Example $Y$ of § 8 is not homogeneous, we made use of the

fact that if $Y$ were homogeneous there would be a homeomorphism of $Y$ onto itself that does not move any point far but which takes one arc component of $Y$ onto another. We formalize this in the following theorem.

THEOREM 8. *If $p$ is a point of a homogeneous, compact, indecomposable, non-degenerate continuum $X$, then there is a sequence of homeomorphisms $h_1, h_2, \ldots$, converging to the identity such that no two $h_i^{-1}(p)$'s belong to the same composant of $X$.*

*Proof.* Let $\{x_\alpha\}$ be an uncountable collection of points all belonging to different composants of $X$ and $h_\alpha$ be a homeomorphism of $X$ onto itself that takes $x_\alpha$ to $p$.

If the collection of homeomorphisms $\{h_\alpha\}$ is metrized by the distance function

$$D(h_\alpha, h_\beta) = \max_{x \epsilon X} \rho(h_\alpha(x), h_\beta(x)),$$

the collection $\{h_\alpha\}$ becomes an uncountable subset of a separable metric function space and some sequence $h_1', h_2', \ldots$, of elements of $\{h_\alpha\}$ converges to an element $h_0'$ of $\{h_\alpha\}$. Then

$$h_i = h_i' h_0'^{-1}.$$

Since $x_1, x_2, \ldots$, belong to different composants of $X$, $h_0'(x_1), h_0'(x_2), \ldots$, also belong to different composants. These are the $h_i^{-1}(p)$'s.

**10. $M'$ contains a folded sequence of arcs.** In this section we complete the proof of Theorem 1. We showed in §§ 2 and 3 that if there is a homogeneous bounded plane continuum $M$ that contains an arc, there is one such $M'$ each of whose proper subcontinua is an arc. Theorem 6 showed that no such $M'$ contains a folded sequence of arcs converging to an arc. Finally, we show that there is no such $M'$ except a circle, for if there were, it would have the following property.

*Property* 20. *$M'$ contains a folded sequence of arcs converging to an arc.* Let $a_0a_6$ be an arc in $M'$ which is accessible from a component of $E^2 - M'$. With no loss of generality we suppose that $a_0a_6$ is horizontal and $a_1, a_2, \ldots, a_5$ are points of $a_0a_6$ such that

$$\text{abscissa } a_i = i \quad (i = 0, 1, \ldots, 6).$$

We suppose furthermore that $a_0a_6$ is accessible from $E^2 - M$ from below. It follows from the methods used in establishing Property 19 that there is a positive number $\epsilon_1$ such that

$$\text{no point of } M' \text{ below } a_0a_6 \text{ is within } \epsilon_1 \text{ of } a_1a_5.$$

Assume $M'$ contains no folded sequence of arcs converging to an arc. Then there is a positive number $\epsilon_2$ such that if $D$ is an $\epsilon_2$-chain covering $a_1a_5$ with

$a_1$ in one end link of $D$ and $a_5$ in the other, then each arc of $M'$ being covered by $D$ and having both ends in the same link of $D$ is of diameter less than $\frac{1}{2}$. Note that $\epsilon_2 < \frac{1}{2}$. Let $D$ be such an $\epsilon_2$ chain covering $a_1a_5$ satisfying conditions of Property 17.

Let $rs$ be an arc such that $rs$ lies above $a_1a_5$; $rs$ misses $M'$; $rs$ is irreducible from the vertical line containing $a_1$ to the vertical line containing $a_5$; the vertical segments $ra_1$, $sa_5$ lie in end links of $D$, and each point between $rs$ and $a_1a_5$ lies in a link of $D$. We find that there is such an $rs$ as follows. Cover $a_0a_6$ by a chain of small mesh satisfying the conditions of Property 17, consider an accessible point of $M'$ above $a_3$ and in one link of this chain, and note from Property 17 and Theorem 6 that this point lies in an accessible arc in $M'$ slightly above $a_0a_6$ and with ends near the ends of $a_0a_6$. It follows from Property 19 that there is an arc in the complement of $M'$ slightly to one side of this first arc. It is this second arc that contains $rs$.

Let $K$ be the topological disc bounded by $a_1a_5$, $a_1r$, $rs$, and $a_5s$. We note that if $p$ is a point of $M'\cdot K$ that is above $a_2a_4$, then the closure of the component of $M'\cdot \operatorname{Int} K$ containing $p$ is an arc irreducible from $a_1r$ to $a_5s$. If it were not, an arc being covered by $D$ and having diameter more than $\frac{1}{2}$ would have ends in the same link of $D$.

Let

$$\epsilon_3 = \min(\epsilon_1, \rho(rs, M')).$$

It follows from Property 18 that there is a positive number $\epsilon_4$ such that if $ab$ is an arc in $M'$ with $\rho(a, b) < \epsilon_4$, then either

diameter $ab < \epsilon_3$, or
$ab$ is $\epsilon_3$ dense in $M'$.

Let $A$ be the arc component of $M'$ containing $a_3$. It follows from Theorem 8 that there is a homeomorphism $h$ of $M'$ onto itself that moves no point by more than $\epsilon_4$ and which takes $a_3$ into a point of $M' - A$. Then $h(a_3)$ is a point of $K$ and lies above $a_2a_4$. Also, $h(a_3)$ lies on an arc in $M'\cdot K$ that is irreducible from $a_1r$ to $a_5s$.

Since $A$ is dense in $M'$, there is an arc $xy$ in $A\cdot K$ such that $xy$ is irreducible from $a_1r$ to $a_5s$ and $xy$ separates $h(a_3)$ from $a_1a_5$ in $K$. By considering points slightly above $a_3$ we find that $xy$ has the following property.

*Special Separating Property*. The arc $xy$ separates two points of $K\cdot(M' - A)$ from each other in $K$ such that the first of the points is above $a_3$ and the other is the image of the first under $h$.

Let $x_1x_2x_3\ldots x_{2n}$ be the arc in $A$ such that $x_1x_2 = xy$, $x_{2n-1}x_{2n} = a_1a_5$, and $x_1x_2$, $x_3x_4$, $x_5x_6$, $\ldots$, $x_{2n-1}x_{2n}$ are the closures of the components of $x_1x_2x_3\ldots x_{2n}\cdot$ $\operatorname{Int} K$ that are irreducible from $a_1r$ to $a_5s$. Then $x_1x_2$ has the special separating property but $x_{2n-1}x_{2n}$ does not.

We now show that if $x_{2i-1}x_{2i}$ has the special separating property, then so does $x_{2i+1}x_{2i+2}$. The resulting contradiction arises as a result of consequences

of our false assumption that there is an $M'$ that contains no folded sequence of arcs converging to an arc.

Suppose $p$, $h(p)$ are points of $K\cdot(M'-A)$ such that $p$ is above $a_3$ and $x_{2i-1}x_{2i}$ separates $p$ from $h(p)$ in $K$. For convenience we suppose that $x_{2i+1}x_{2i+2}$ is below $x_{2i-1}x_{2i}$ and $h(p)$ is above $x_{2i-1}x_{2i}$. (Other cases are handled with similar arguments to that given in this case.) Then $x_{2i+1}x_{2i+2}$ separates $p$ from $h(p)$ in $K$ unless $p$ is above $x_{2i+1}x_{2i+2}$, so we suppose $p$ is between $x_{2i-1}x_{2i}$ and $x_{2i+1}x_{2i+2}$. Our proof now breaks down into two cases.

*Case* 1. *If $x_{2i}$, $x_{2i+1}$ belong to the same one of $a_1r$, $a_5s$,* (see Figure 2). Let $tu$ be the closure of the component of $M'\cdot\text{Int }K$ containing $p$. It is an arc irreducible from $a_1r$ to $a_5s$. We suppose $u$ belongs to the vertical line through $a_1$ containing $x_{2i}$ and $x_{2i+1}$.

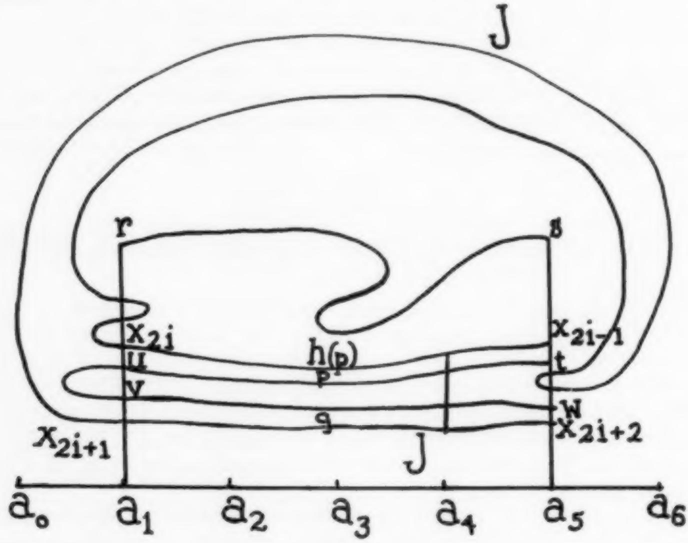


FIGURE 2

Suppose a point moves in an arc in $M'$ through $p$, past $u$, and $(vw)$ is the next component of $M'\cdot\text{Int }K$ it meets whose closure $vw$ is an arc irreducible from $a_1r$ to $a_5s$. Let $q$ be a point of $vw$ directly above $a_3$. It follows from the Jordan curve theorem that $q$ lies between $x_{2i-1}x_{2i}$ and $x_{2i+1}x_{2i+2}$. Also, $q$ is below $tu$ and $\rho(q, x_{2i-1}x_{2i}) > \epsilon_4$ or else the arc $tuvw$ contains an arc $ab$ such that

$\rho(a, b) < \epsilon_4$,
diameter $ab > \epsilon_3$, and
$\rho(ab, a_4) > \epsilon_3$.

If $q$ were above $tu$, we would take $p = a$ and let $b$ be a point of $vw$ between $p$ and $x_{2i-1}x_{2i}$. If $q$ were below $tu$ and $\rho(q, x_{2i-1}x_{2i}) < \epsilon_4$, we would take $q$ to be $b$ and $a$ to be a point of $tu$ between $q$ and $x_{2i-1}x_{2i}$. Since the existence of such an arc $ab$ violates the definition of $\epsilon_4$, we suppose that

$$\rho(q, x_{2i-1}x_{2i}) > \epsilon_4.$$

We now show that $x_{2i+1}x_{2i+2}$ separates $q$ from $h(q)$ in $K$. Note that $q$ is above $x_{2i+1}x_{2i+2}$.

Consider the simple closed curve $J$ that is the sum of a vertical interval in $K$ above $a_4$ and an arc in $x_{2i-1}x_{2i+2}$ that contains $x_{2i}x_{2i+1}$. Note that no point of the arc $pq$ in $M$ is within $\epsilon_4$ of this vertical part of $J$ above $a_4$. As a point moves from $p$ to $q$, the image of the point under $h$ does not intersect $J$. Hence, $h(q)$ is either above $x_{2i-1}x_{2i}$ or below $x_{2i+1}x_{2i+2}$. It is not above $x_{2i-1}x_{2i}$ because $\rho(q, x_{2i-1}x_{2i}) > \epsilon_4$ and $\rho(q, h(q)) < \epsilon_4$. Hence $x_{2i+1}x_{2i+2}$ has the special separation property.

*Case* 2. *If $x_{2i}$ and $x_{2i+1}$ belong to different vertical lines, see Figure* 3. We suppose $x_{2i}$, $u$, $x_{2i+2}$ belong to $a_5s$ and define $vw$ and $q$ as in Case 1.

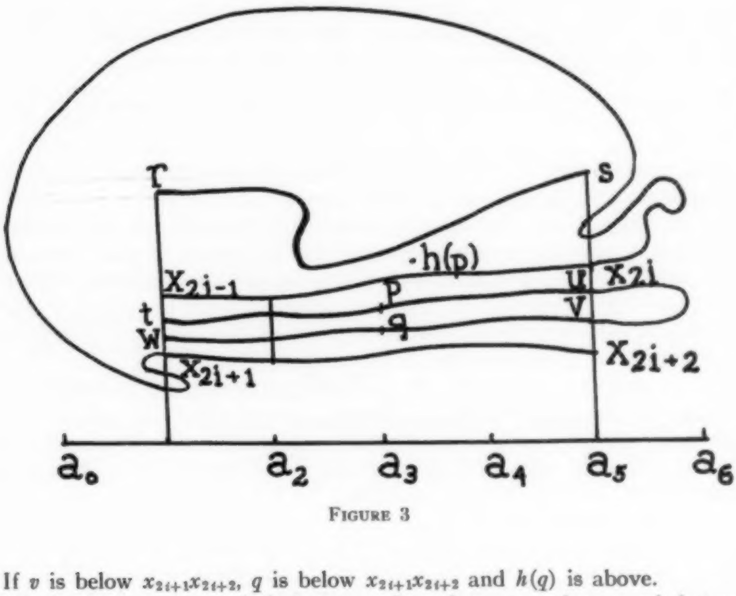


FIGURE 3

If $v$ is below $x_{2i+1}x_{2i+2}$, $q$ is below $x_{2i+1}x_{2i+2}$ and $h(q)$ is above.

If $v$ is above $x_{2i+1}x_{2i+2}$ it is between the points $x_{2i}$ and $x_{2i+2}$, $q$ is between $x_{2i-1}x_{2i}$ and $x_{2i+1}x_{2i+2}$, and $h(q)$ is above $x_{2i-1}x_{2i}$. Since each of $p$, $q$ is within $\epsilon_4$ of $x_{2i-1}x_{2i}$, it follows as in Case 1 that $tuvw$ contains an arc $ab$ such that

$$\rho(a, b) < \epsilon_4,$$
$$\text{diameter } ab > \epsilon_3,$$
$$\rho(ab, a_2) > \epsilon_3.$$

These conditions violate the definition of $\epsilon_4$.

We note that in establishing Property 20 we used properties of the plane or 2-sphere and not just properties of an arbitrary 2-manifold alone. If this could be by-passed, one might get an affirmative answer to the following question.

*Question.* Suppose $X$ is a 1-dimensional homogeneous compact continuum that contains an arc and lies on a compact 2-manifold. Is $X$ necessarily a simple closed curve?

**11. Circle-like continua and tree-like continua.** Solenoids and a circle of pseudo-arcs are the known examples of homogeneous circle-like continua. Since each proper subcontinuum of a circle-like continuum is snake-like and each homogeneous non-degenerate snake-like continuum is a pseudo-arc, one might suspect that the answer to the following is in the affirmative.

*Question.* Does each homogeneous circle-like continuum other than a solenoid contain a pseudo-arc? We do not provide an answer.

THEOREM 9. *Each homogeneous circle-like continuum that contains an arc is a solenoid.*

*Indication of proof.* This theorem is much easier to establish than Theorem 1 but the same method of attack may be used.

By using rays as in § 3, it may be shown that the homogeneous circle-like continuum $X$ contains a non-degenerate subcontinuum $X'$ such that each proper subcontinuum of $X$ is an arc. In proving the counterpart of Property 9, we cannot use Theorem 3 (which is a theorem about the plane) to show that $ab + \bar{R}$ cannot lie in $X$, but instead we use the fact that each proper subcontinuum of $X$ is snake-like to prove this.

We may as well suppose that $X' = X$, for if it is not, it is snake-like, it is a pseudo-arc **(5)**, and it contains no arc.

We finish the indication of proof of Theorem 9 by showing that there is a sequence of circular chains (open coverings whose 1-nerves are simple closed curves) $D_1, D_2, \ldots$, covering $X$ such that

1. $D_{i+1}$ is a refinement of $D_i$,
2. $D_{i+1}$ circles around $D_i n_i$ times without any folding back, and
3. the mesh of $D_{i+1}$ is less than $1/2^i$ times the distance between any two non-adjacent elements of $D_i$.

It is then only a matter of getting an open covering of a similar kind of the solenoid which is the intersection of the tori $T_1, T_2, \ldots$, where $T_{i+1}$

winds about $T_i$ $n_i$ times and use the two coverings to get a homeomorphism of $X$ onto the solenoid. ($D_{i+1}$ is said to fold back in $D_i$ if $D_i$ contains two adjacent links $d_x$, $d_y$, and $D_{i+1}$ contains a subchain $E$ such that each link of $E$ lies in either $d_x$ or $d_y$, each end link of $E$ intersects $d_x - d_y$ but not each link of $E$ lies in $d_x$.)

Suppose that $D_i$ has already been obtained and it is such that there is a positive number $\epsilon$ such that if $D'$ is any circular chain of mesh less than $\epsilon$ covering $X$, then $D'$ refines $D_i$ and circles about $D_i$ without any folding back. We show how to get $D_{i+1}$. With no loss of generality we suppose that $\epsilon < 1/2^i$ times the distance between any two non-adjacent elements of $D_i$.

We apply Theorem 6 to show that no folded sequence of arcs in $X$ converges to an arc. Hence, there is a $\delta$ such that if $ab$ is an arc of diameter greater than $\epsilon/14$, no $\delta$ chain $D''$ covers $ab$ in such a way that both $a$, $b$ lie in the same link of $D''$. We suppose $\delta < \epsilon/14$.

Let $D$ be a $\delta$ circular chain covering $X$ with its links ordered $d_1, d_2, \ldots, d_n$. Let $d_{n_1} = d_1$; $d_{n_2}$ is the first link of $D$ whose distance from $d_{n_1}$ is more than $\epsilon/14$; $d_{n_3}$ is the next link of $D$ after $d_{n_2}$ whose distance from $d_{n_2}$ is more than $\epsilon/14 \ldots$. Let $d_{n_{2r-1}}$ or $d_{n_{2r}}$ be the last such link obtained.

The first link of $D_{i+1}$ is the sum of the links between $d_{n_1}$ and $d_{n_4}$ inclusive; the next link of $D_{i+1}$ is the sum of the links of $D$ between $d_{n_3}$ and $d_{n_6}$ inclusive; $\ldots$; and the last link of $D_{i+1}$ is the sum of the links between $d_{n_{2r-1}}$ and $d_{n_2}$ inclusive (this link contains $d_n$ and $d_1$). Each link of $D_{i+1}$ other than the last is of diameter less than $4\epsilon/14 + 7\delta$ and the last is of diameter less than $5\epsilon/14 + 9\delta$. In either case, $D$ is of mesh less than $\epsilon$. If $D'$ is a refinement of $D_{i+1}$ of mesh less than $\delta$, $D'$ circles about $D_{i+1}$ without any folding back.

A triodic continuum is the sum of three continua $A$, $B$, $C$ such that $A \cdot B = A \cdot C = B \cdot C$ is a proper subcontinuum of each of $A$, $B$, $C$. Theorem 7 did not provide an answer as to whether or not each homogeneous tree-like continuum fails to contain an arc. Our methods do not give this because we fail to prove the counterpart of Properties 4 and 9.

THEOREM 10. *A homogeneous tree-like continuum contains no arc if it contains no triodic continuum.*

To establish Theorem 10 we use the hypothesis that the continuum contains no triodic continuum to establish the counterpart of Property 9. Property 15 then follows and reduces Theorem 10 to Theorem 7.

## References


1. R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. Math., *67* (1958), 313–324.
2. R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J., *15* (1948), 729–742.
3. ——— *Snake-like continua*, Duke Math. J., *18* (1951), 653–663.
4. ——— *Concerning hereditarily indecomposable continua*, Pac. J. Math., *1* (1951), 43–51.
5. ——— *Each homogeneous chainable continuum is a pseudo arc*, Proc. Amer. Math. Soc., *10* (1959), 345–346.
6. R. H. Bing and F. B. Jones, *Another homogeneous plane continuum*, Trans. Amer. Math. Soc., *90* (1959), 171–192.
7. C. E. Burgess, *Continua and various types of homogeneity*, Trans. Amer. Math. Soc., *88* (1958), 366–374.
8. H. J. Cohen, *Some results concerning homogeneous plane continua*, Duke Math. J., *18* (1951), 467–474.
9. E. E. Grace, *Totally nonconnected im kleinen continua*, Proc. Amer. Math. Soc., *9* (1958), 818–821.
10. F. B. Jones, *Certain homogeneous unicoherent indecomposable continua*, Proc. Amer. Math. Soc., *2* (1951), 855–859.
11. S. Mazurkiewicz, *Sur les continus homogènes*, Fund. Math., *5* (1924), 137–146.
12. E. E. Moise, *A note on the pseudo-arc*, Trans. Amer. Math. Soc., *64* (1949), 57–58.
13. R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Coll. Publ., *13* (1932).


*University of Wisconsin*

# SUR LA COHOMOLOGIE NON ABELIENNE I (dimension deux)

PAUL DEDECKER

**1. Introduction.** On sait que la définition de la cohomologie d'un espace $X$ à coefficients dans un faisceau $\mathfrak{G}$ de groupes non abéliens est liée à la classification de certains types d'espaces fibrés principaux de base $X$ (**1**; **8**; **10**). Cette cohomologie se définit assez facilement en dimensions zéro et un, mais pour certains problèmes, il serait utile de pouvoir la définir en dimension deux également.

J'ai abordé la question dans (**3**; **6**) mais un certain nombre de difficultés subsistaient qu'il semble utile de chercher à éliminer, ce qui est l'objet de la présente note.

Remarquons d'abord que la définition de $H^0(X, \mathfrak{G})$, groupe des sections de $\mathfrak{G}$, est triviale, aucune difficulté ne résultant du caractère non abélien de $\mathfrak{G}$. En ce qui concerne $H^1(X, \mathfrak{G})$ une difficulté naît du fait que cet ensemble n'est plus un groupe, même non abélien; il possède toutefois un élément neutre privilégié. Par contre on ne voit pas du tout comment définir un $H^2(X, \mathfrak{G})$; plus précisément on ne sait pas comment caractériser un 2-cocycle parmi les 2-cochaînes, ni *a fortiori* les classes de 2-cocycles qui constitueraient des classes de 2-cohomologie. Une solution avait été proposée dans (**3**) en appelant 2-cocycle une 2-cochaîne qui devenait un cobord par un plongement convenable du faisceau $\mathfrak{G}$ dans un plus grand faisceau $\mathfrak{H}$ et par une définition des classes destinées à prolonger jusqu'en dimension deux la suite exacte de cohomologie associée à une suite exacte de coefficients. Cette solution n'était qu'imparfaite pour deux raisons: 1. la cohomologie $H^2$ obtenue était tronquée, c'est-à-dire ne redonnait qu'une partie de la cohomologie usuelle dans le cas abélien; 2. les classes de 2-cohomologie n'étaient plus des classes d'équivalence. (La version proposés dans (**6**) éliminait cette dernière difficulté mais pas la première.)

Par après, j'ai remarqué que l'on pouvait récupérer une structure algébrique en dimension un à condition de plonger l'ensemble $H^1(X, \mathfrak{G})$ dans un *groupoïde* $\mathfrak{H}^1(X, \mathfrak{G})$ se réduisant au groupe de cohomologie usuel dans le cas où $\mathfrak{G}$ est abélien et possédant par ailleurs une signification géométrique intéressante. Toutefois afin de conserver la technique des suites exactes il devenait nécessaire de plonger les $H^0(X, \mathfrak{G})$ dans de nouveaux ensembles $\mathfrak{H}^0(X, \mathfrak{G})$ munis également d'une structure de groupoïde adéquate (et non nécessairement unique).

Il est assez remarquable qu l'utilisation de groupoïdes en dimension un permet de donner une solution aux difficultés rencontrées pour définir un

Reçu le 21 Novembre, 1958.

$H^2(X, \mathfrak{G})$. C'est cette solution que nous exposons ici sans nous occuper de récupérer une structure algébrique. Celle-ci demanderait qu'il soit fait appel à ce que nous avons appelé des "espaces fibrés non holonomes" **(6)** et impliquerait en outre un nouvel élargissement des groupoïdes $\mathfrak{H}^1$ et $\mathfrak{H}^0$.

**2. Système $\Phi$ de coefficients non abéliens.** Rappelons qu'un *faisceau de groupes* sur un espace topologique $X$ est un nouvel espace $\mathfrak{G}$ satisfaisant aux conditions suivantes:

(1) il existe une application continue $p$ de $\mathfrak{G}$ sur $X$ appelée *projection*;

(2) tout point $g \in \mathfrak{G}$ possède un voisinage ouvert $U$ tel que la restriction $p|U$ soit un homéomorphisme de $U$ sur un ouvert de $X$;

(3) chaque $\mathfrak{G}_x = p^{-1}(x)$, $x \in X$, est muni d'une structure de groupe d'unité $e_x$ telle que les applications $x \to e_x$, $g \to g^{-1}$, $(g, h) \to g.h$ (respectivement de $X$ dans $\mathfrak{G}$, de $\mathfrak{G}$ dans $\mathfrak{G}$, de $\mathfrak{G} \times_X \mathfrak{G}$ dans $\mathfrak{G}$) soient continues. (On note $\mathfrak{G} \times_X \mathfrak{G}$ l'ensemble des couples $(g, h) \in \mathfrak{G} \times \mathfrak{G}$ tels que $p(g) = p(h)$.)

Les ensembles $\mathfrak{G}_x$, $x \in X$ sont appelés les *tiges* du faisceau $\mathfrak{G}$.

Pour tout groupe $G$, notons $I(G)$ le groupe de ses automorphismes intérieurs, quotient de $G$ par son centre $Z(G)$. Si $\mathfrak{G}$ est un faisceau de groupes, soit $\mathfrak{Y}$ l'intérieur de la réunion des $Z(\mathfrak{G}_X)$. Le quotient $\mathfrak{G}/\mathfrak{Y}$ est un nouveau faisceau noté $\mathfrak{J}(\mathfrak{G})$ et appelé faisceau des automorphismes intérieurs de $\mathfrak{G}$. On a pour tout $x$ une surjection canonique $i_x$ et une surjection globale $i$:

$$i_x: \mathfrak{G}_x \to \mathfrak{J}_x, \qquad i: \mathfrak{G} \to \mathfrak{J}.$$

La topologie de $\mathfrak{J}$ est définie de telle sorte qu'une section locale $s$: $U \to \mathfrak{J}$ ($U$ ouvert de $X$) soit, dans un voisinage $V$ de tout point $x \in U$ de la forme $s = it$ où $t$: $V \to \mathfrak{G}$ est une section locale de $\mathfrak{G}$ (topologie telle que $i$ soit un homomorphisme de faisceaux).

N.B. Par "section" on entend "section continue." Par "application locale" définie sur un espace $X$, on entend une application définie sur un ouvert de $X$.

Soit maintenant $\mathfrak{A}$ un nouveau faisceau de groupes sur $X$ qui soit un faisceau de groupes d'opérateurs sur $\mathfrak{G}$ et supposons donné un homomorphisme

$$\rho: \mathfrak{G} \to \mathfrak{A}$$

tel que

(2.1) pour tout $g \in \mathfrak{G}_x$, $\rho(g)$ opère comme l'automorphisme intérieur défini par $g$: $\rho(g)\gamma = g\gamma g^{-1}$, $\gamma \in \mathfrak{G}_x$;

(2.2) si $\alpha \in \mathfrak{A}_x$ et $g \in \mathfrak{G}_x$, on a (dans $\mathfrak{A}_x$ la multiplication y étant notée $\circ$)

$$\rho[\alpha(g)] \circ \alpha = \alpha \circ \rho(g) \qquad \text{ou} \qquad \rho[\alpha(g)] = \alpha \circ \rho(g) \circ \alpha^{-1}.$$

*Exemple.* Les conditions précédentes sont par exemple vérifiées si $\mathfrak{G}$ est un sous faisceau normal d'un faisceau $\mathfrak{H}$ de groupes sur $X$ et si $\mathfrak{A} = \mathfrak{J}(\mathfrak{G})$, $\rho$ étant induit par l'inclusion $\mathfrak{G} \to \mathfrak{H}$. Ou encore si $\mathfrak{A}$ est un faisceau de groupes d'automorphismes de $\mathfrak{G}$ contenant $\mathfrak{J}(\mathfrak{G})$ et $\mathfrak{G} \to \mathfrak{A}$ l'application canonique.

*Définition* 2.1. Un triple $\Phi = (\mathfrak{G}, \rho, \mathfrak{A})$ satisfaisant à ces conditions sera appelé un *système de coefficients pour la cohomologie* (non abélienne).

*Remarque* 2.1. Un simple faisceau $\mathfrak{G}$ contient suffisamment d'informations pour construire un *groupe* de cohomologie de dimension zéro et un *ensemble* ou un *groupoïde* de cohomologie de dimension un. Toutefois dans ce groupoïde un faisceau d'automorphismes intervient de façon cachée et il doit être explicité si l'on veut pouvoir écrire des suites exactes de groupoïdes.

*Remarque* 2.2. Le cas usuel en cohomologie—que nous appelerons dans la suite le *cas des coefficients abéliens*—est celui où $\mathfrak{G}$ est un faisceau de groupes abéliens et où $\mathfrak{A}$ est le faisceau trivial tel que le groupe $\mathfrak{A}_x$ pour tout $x \in X$ soit réduit au seul élément unité et où $\rho$ est l'homomorphisme trivial. Bien entendu si $\mathfrak{A}$ est différent de ce faisceau trivial, $\mathfrak{G}$ étant cependant abélien, on ne se trouve déjà plus dans le "cas abélien."

Afin d'être complets, rappelons encore la définition d'un groupoïde, cas particulier des systèmes multiplicatifs.

*Definition* 2.2. On appelle *système multiplicatif* un ensemble muni d'une loi de composition non partout définie appelée multiplication. Dans un tel système on appelle *unité* tout élément $e$ tel que si un composé $ex$ ou $xe$ est défini, ce composé est toujours égal à $x$. Deux unités composables sont donc égales.

*Definition* 2.3. On appelle *groupoïde* un système multiplicatif satisfaisant aux propriétés suivantes:

$G1$. Si l'un des éléments $x(yz)$ ou $(xy)z$ est défini, l'autre l'est et ils sont égaux;

$G2$. à tout $x$ correspondent des unités $e_x$ et ${}_xe$ telles que les composés $x\, e_x$ et ${}_xe\, x$ soient définis;

$G3$. à tout $x$ correspond un $x'$ tel que $x'x$ soit défini et égal à $e_x$.

*Propriétés des groupoïdes*:

1. Les unités $e_x$ et ${}_xe$ sont uniquement définies par $x$.
2. Un composé $xy$ est défini si et seulement si $e_x = {}_ye$.
3. Une unité est toujours composable avec elle-même.
4. Le composé $xx'$ est toujours défini et vaut ${}_xe$.
5. Une égalité $yx = zx$ (resp. $xy = xz$) entraîne $y = z$; donc $x'$ est uniquement défini par $x$; ou le note $x^{-1}$.

**3. Cochaînes à valeurs dans** $\Phi$. Le systéme $\Phi$ étant choisi une fois pour toutes, considérons un recouvrement ouvert $\mathfrak{U} = (U_i)_{i \in I}$ de $X$. Pour une suite $i_1, \ldots, i_k$ d'indices, on posera comme d'habitude

$$U_{i_1, \ldots, i_k} = U_{i_1} \cap \ldots \cap U_{i_k}.$$

Considérons les couples $(g_{ij}, \alpha_{ij})$ de sections locales

$$g_{ij}: U_{ij} \to \mathfrak{G}, \qquad \alpha_{ij}: U_{ij} \to \mathfrak{A}.$$

*Définition* 3.1. Ces couples sont appelés 1-*cochaînes* (de $\mathfrak{U}$ à valeurs dans $\Phi$); on vérifie qu'ils forment un groupoïde $\mathfrak{C}^1(\mathfrak{U}, \Phi)$ si l'on adopte une loi de composition pour laquelle un produit

$$(g'_{ij}, \alpha'_{ij})\cdot(g_{ij}, \alpha_{ij}) \tag{3.2}$$

est défini si $\alpha'_{ij} = \rho(g_{ij})\alpha_{ij}$ et est alors égal à

$$(g'_{ij}g_{ij}, \alpha_{ij}). \tag{3.3}$$

*Définition* 3.4. La cochaîne $(g_{ij}, \alpha_{ij})$ est dite *alternée* si

$$\alpha_{ij} = \alpha_{ji}^{-1}, \quad \alpha_{ii} = 1, \tag{3.5}$$

$$g_{ij} = \alpha_{ij}(g_{ji}^{-1}), \quad g_{ii} = 1. \tag{3.6}$$

PROPOSITION 3.7. *Les cochaînes alternées forment une sous-groupoïde* $\mathfrak{C}_a^1(\mathfrak{U}, \Phi)$ *de* $\mathfrak{C}^1(\mathfrak{U}, \Phi)$.

*Démonstration*. Supposons le produit (3.2) défini, ses facteurs étant des cochaînes alternées; il suffit de montrer que sa valeur (3.3) vérifie

$$g'_{ij}\cdot g_{ij} = \alpha_{ij}(g_{ji}^{-1}\cdot g'^{-1}_{ji}).$$

Or on a

$$g_{ij} = \alpha_{ij}(g_{ji}^{-1}),$$
$$g'_{ij} = \alpha'_{ij}(g'^{-1}_{ji}) = g_{ij}\cdot\alpha_{ij}(g'^{-1}_{ji})|g_{ij}^{-1},$$
$$g'_{ij}\cdot g_{ij} = g_{ij}\cdot\alpha_{ij}(g'^{-1}_{ji})\cdot g_{ij}^{-1}\cdot g_{ij} = \alpha_{ij}(g_{ji}^{-1}g'^{-1}_{ji}).$$

C.q.f.d.

## 4. Cochaînes de dimension deux.

*Définition* 4.1. Une 2-*cochaîne* de $\mathfrak{U}$ à valeurs dans $\Phi$ est un triple $(\alpha_{ij}', \gamma_{ijk}, \alpha_{ij})$ de sections locales

$$\alpha_{ij}, \alpha'_{ij}: U_{ij} \to \mathfrak{A}, \qquad \gamma_{ijk}: U_{ijk} \to \mathfrak{G}$$

tel que dans $U_{ijk}$ on ait

$$\alpha'_{ij}\alpha'_{jk}\alpha'_{ki} = \rho(\gamma_{ijk})\alpha_{ij}\alpha_{jk}\alpha_{ki}. \tag{4.2}$$

Ces cochaînes forment un groupoïde $\mathfrak{C}^2(\mathfrak{U}, \Phi)$ si l'on convient qu'un produit

$$(\beta'_{ij}, \theta_{ijk}, \beta_{ij})(\alpha'_{ij}, \gamma_{ijk}, \alpha_{ij})$$

est défini si et seulement si $\beta_{ij} = \alpha'_{ij}$ et est alors égal à

$$(\beta'_{ij}, \theta_{ijk}\cdot\gamma_{ijk}, \alpha_{ij}).$$

Cette définition est légitime car si les deux facteurs sont des cochaînes il en est de même du produit.

Pour toute 1-cochaîne $(g, \alpha) = (g_{ij}, \alpha_{ij})$ on posera

$$(g\alpha)_{ij} = \rho(g_{ij}) \circ \alpha_{ij},$$
$$g^{i}_{ij} = g_{ij} \text{ défini dans } U_{ij},\ g^{ij}_{jk} = \alpha_{ij}(g_{jk}) \qquad \text{défini dans } U_{ijk},$$
$$g^{ijk}_{kl} = \alpha_{ij} \circ \alpha_{jk}(g_{kl}), \qquad \text{défini dans } U_{ijkl}, \text{ etc. . . .}$$

(4.3) $$(\delta_\alpha g)_{ijk} = g^{i}_{ij} \cdot g^{ij}_{jk} \cdot g^{ijk}_{ki}.$$

On remarquera que la première des conditions (3.6) de l'alternation d'une 1-cochaîne $(g, \alpha)$ s'écrit

$$g^{-1}_{ij} = (g^{ij}_{ji}).$$

*Définition* 4.2. On appelle 1-*cobord* l'opérateur

$$\delta^1 \colon \mathfrak{C}^1(\mathfrak{U}, \Phi) \to \mathfrak{C}^2(\mathfrak{U}, \Phi)$$

défini par

(4.4) $$\delta^1(g, \alpha) = (g\alpha, \delta_\alpha g, \alpha) \quad \text{ou explicitement}$$
$$\delta^1(g_{ij}, \alpha_{ij}) = ((g\alpha)_{ij}, (\delta_\alpha g)_{ijk}, \alpha_{ij}).$$

Il faut évidemment vérifier que la condition (4.2) est bien remplie par (4.4), ce qui résulte de

$$\begin{aligned}(g\alpha)_{ij}(g\alpha)_{jk}(g\alpha)_{ki} &= \rho(g_{ij}) \circ \alpha_{ij} \circ \rho(g_{jk}) \circ \alpha_{jk} \circ \rho(g_{ki}) \circ \alpha_{ki} \\ &= \rho(g_{ij}) \circ [\alpha_{ij}\rho(g_{jk})\alpha^{-1}_{ij}] \circ [\alpha_{ij}\alpha_{jk}\rho(g_{ki})\alpha^{-1}_{jk}\alpha^{-1}_{ij}] \circ (\alpha_{ij} \circ \alpha_{jk} \circ \alpha_{ki}) \\ &= \rho(g_{ij}) \circ \rho(g^{ij}_{jk}) \circ \rho(g^{ijk}_{ki}) \circ (\alpha_{ij} \circ \alpha_{jk} \circ \alpha_{ki}) \\ &= \rho(g_{ij} g^{ij}_{jk} g^{ijk}_{ki}) \circ (\alpha_{ij} \circ \alpha_{jk} \circ \alpha_{ki}).\end{aligned}$$

Le passage de la deuxième à la troisième ligne résulte de la condition (2.2). C.q.f.d.

*Définition* 4.3. On appelle *fondamentale* une 1-cochaîne $(g, \alpha)$ ou une 2-cochaîne $(\alpha', \gamma, \alpha)$ telle que les sections $\alpha_{ij}$ coïncident avec les sections unitaires $\epsilon_{ij}$ qui associent à tout $x \in U_{ij}$ l'unité de $\mathfrak{A}_x$. Ces cochaînes seront souvent notées $g = (g_{ij})$, $(\alpha', \gamma) = (\alpha_{ij}, \gamma_{ijk})$; elles forment des ensembles que l'on notera $C^1(\mathfrak{U}, \Phi) = C^1(\mathfrak{U}, \mathfrak{G})$ et $C^1(\mathfrak{U}, \Phi)$ (avec l'indice $a$ en cas d'alternation).

*Remarque*. Le cobord d'une 1-cochaîne fondamentale $g = (g_{ij})$ est donné par

(4.5) $$\delta g = (\rho(g_{ij}), g_{ij}g_{jk}g_{ki}).$$

On posera

(4.6) $$(\delta g)_{ijk} = g_{ij}g_{jk}g_{ki}.$$

En multipliant une cochaîne fondamentale à gauche par une cochaîne quelconque telle que le produit soit défini, on obtient toujours une cochaîne fondamentale. Il est clair que le cobord d'une 1-cochaîne fondamentale est une 2-cochaîne fondamentale.

**Proposition** 4.4. *Le cobord $\delta^1$ est un homomorphisme de groupoïdes.*

*Démonstration.* Si le produit $(g', \alpha')\cdot(g, \alpha)$ est défini, on a $\alpha' = g\alpha$ et le produit $\delta(g', \alpha')$. $\delta(g, \alpha)$ est donc lui-même défini. Il reste à vérifier que

$$\delta(g'g, \alpha) = \delta(g', \alpha')\cdot\delta(g', \alpha).$$

Le second membre vaut

$$((g'\alpha')_{ij}, (\delta_{\alpha'}g')_{ijk}, \alpha'_{ij})(\alpha'_{ij}, (\delta_\alpha g)_{ijk}, \alpha_{ij}) = ((g'\alpha')_{ij}, (\delta_{\alpha'}g')_{ijk}(\delta_\alpha g)_{ijk}, \alpha_{ij})$$

où $\alpha'_{ij} = (gd)_{ij}$. On a $g'\alpha' = g'(g\alpha) = (g'g)\alpha$ et la propriété résulte de

$$\begin{aligned}
(\delta_{\alpha'}g')_{ijk}\cdot(\delta_\alpha g)_{ijk}
&= g'_{ij}\cdot\alpha'_{ij}(g'_{jk})\cdot\alpha'_{ij}\alpha'_{jk}(g'_{ki})\cdot[g_{ij}\cdot\alpha_{ij}(g_{jk})\cdot\alpha_{ij}\alpha_{jk}(g_{ki})]\\
&= g'_{ij}\cdot g_{ij}\cdot\alpha_{ij}(g'_{jk})\cdot \overset{-1}{g_{ij}}\cdot g_{ij}\cdot\alpha_{ij}[g_{jk}\cdot\alpha_{jk}(g'_{ki})\cdot\overset{-1}{g_{jk}}]\cdot\overset{-1}{g_{ij}}\cdot[\ldots]\\
&= (g'_{ij}\cdot g_{ij})\cdot\alpha_{ij}(g'_{jk}\cdot g_{jk})\cdot\alpha_{ij}\alpha_{jk}(g'_{ki})[\alpha_{ij}(\overset{-1}{g_{jk}})\cdot\alpha_{ij}(g_{jk})]\alpha_{ij}\alpha_{jk}(g_{ki})\\
&= (g'g)_{ij}\cdot\alpha_{ij}[(g'g)_{jk}]\cdot\alpha_{ij}\alpha_{jk}[(g'g)_{ki}]\\
&= (\delta_\alpha g'g)_{ijk}.
\end{aligned}$$

C.q.f.d.

## 5. Alternation en dimension deux.

**Proposition** 5.1. *Soit $(\alpha_{ij}, \gamma_{ijk}, \epsilon_{ij})$ le cobord d'une 1-cochaîne alternée fondamentale $(g_{ij}, \epsilon_{ij})$. Dans ces conditions $\alpha_{ij}$ est alternée et on a les identités*

$$\begin{aligned}
&\gamma_{jki} = \alpha_{jk}\alpha_{ki}(\gamma_{ijk}) &&\text{ou}\quad \gamma_{ijk} = \alpha_{ik}\alpha_{kj}(\gamma_{jki})\\
&\gamma_{kij} = \alpha_{ki}(\gamma_{ijk}) &&\text{ou}\quad \gamma_{ijk} = \alpha_{ik}(\gamma_{kij})\\
&\gamma_{ikj} = \overset{-1}{\gamma_{ijk}} &&\\
&\gamma_{kji} = \alpha_{kj}\alpha_{ji}(\overset{-1}{\gamma_{ijk}}) &&\text{ou}\quad \overset{-1}{\gamma_{ijk}} = \alpha_{ij}\alpha_{jk}(\gamma_{kji})\\
&\gamma_{jik} = \alpha_{ji}(\overset{-1}{\gamma_{ijk}}) &&\text{ou}\quad \overset{-1}{\gamma_{ijk}} = \alpha_{ij}(\gamma_{jik}).
\end{aligned}$$

*Démonstration.* C'est une conséquence immédiate des relations

$$\alpha_{ij} = \rho(g_{ij}),\quad \gamma_{ijk} = g_{ij}g_{jk}g_{ki},\quad g_{ij} = \overset{-1}{g_{ji}}.$$

Par exemple la première relation découle de

$$\gamma_{jki} = g_{jk}g_{ki}g_{ij} = g_{jk}g_{ki}(g_{ij}g_{jk}g_{ki})g_{ik}g_{kj} = \alpha_{jk}\alpha_{ki}(\gamma_{ijk}).$$

De même $\gamma_{ijk} = \gamma_{ikj}{}^{-1}$ résulte de

$$\gamma_{ijk} = g_{ij}g_{jk}g_{ki} = (g_{ik}g_{kj}g_{ji})^{-1} = \overset{-1}{\gamma_{ikj}}.$$

*Définition* 5.2. Une 2-cochaîne fondamentale $(\alpha_{ij}, \gamma_{ijk}, \epsilon_{ij})$ est dite *alternée* si $\alpha_{ij}$ est alternée (c'est-à-dire vérifie la condition (3.5) de la définition 3.4) et si elle satisfait aux identités précédentes.

**Proposition 5.3.** *Pour qu'une 2-cochaîne fondamentale* $(\alpha_{ij}, \gamma_{ijk}, \epsilon_{ij})$ *soit alternée, il faut et il suffit que* $\alpha_{ij}$ *le soit et que l'on ait pour tout triple* $(i, j, k)$ *d'indices:*

$$\gamma_{kij} = \alpha_{ki}(\gamma_{ijk}) \quad \text{et} \quad \gamma_{ikj} = \gamma_{ijk}^{-1}. \tag{5.1}$$

En effet de ces identités résultent

$$\gamma_{jki} = \alpha_{jk}(\gamma_{kij}) = \alpha_{jk}(\alpha_{ki}(\gamma_{ijk})) = \alpha_{jk}\alpha_{ki}(\gamma_{ijk}),$$
$$\gamma_{jik} = \alpha_{ji}(\gamma_{ikj}) = \alpha_{ji}(\gamma_{ijk}^{-1}),$$
$$\gamma_{kji} = \alpha_{kj}\alpha_{ji}(\gamma_{ikj}) = \alpha_{kj}\alpha_{ji}(\gamma_{ijk}^{-1}).$$

C.q.f.d.

## 6. 2-cocycles.

**Proposition 6.1.** *Si la 2-cochaîne alternée fondamentale* $(\alpha_{ij}, \gamma_{ijk}, \epsilon_{ij})$ *est le cobord de la 1-cochaîne alternée fondamentale* $(g_{ij}, \epsilon_{ij})$, *elle vérifie pour tout quadruple ordonné* $(i, j, k, l)$ *la condition*

$$\gamma_{ljk} = \gamma_{lji}\cdot\alpha_{li}(\gamma_{ijk})\cdot\gamma_{lik} \qquad (\text{dans } U_{ijkl}). \tag{6.2}$$

*Démonstration*. En effet le second membre vaut

$$(g_{lj}g_{ji}g_{il})\cdot g_{li}\cdot(g_{ij}g_{jk}g_{ki})\cdot g_{il}\cdot(g_{li}g_{ik}g_{kl}) = g_{lj}g_{jk}g_{kl}.$$

C.q.f.d.

*Définition* 6.2. Une 2-cochaîne fondamentale alternée est appelée *cocycle* si elle vérifie la condition (6.2) dans $U_{ijkl}$ pour tout quadruple ordonné d'indices $(i, j, k, l)$.

La proposition 6.1 signifie que tout cobord d'une 1-cochaîne fondamentale alternée est un 2-cocycle.

Rappelons que l'on dit qu'*un groupoïde* $\Gamma$ *opère à gauche* sur un ensemble $E$ si l'on s'est donné une loi de composition (non partout définie) que l'on notera multiplicativement à gauche $\Gamma \times E \rightarrow E$ satisfaisant aux conditions suivantes:

1. si l'un des éléments

$$(g\cdot g')\cdot x, \qquad g\cdot(g'x), \qquad g, g' \in \Gamma, \qquad x \in E$$

est défini l'autre l'est aussi et ces éléments sont égaux;

2. si $e$ est une unité de $\Gamma$, chaque fois que $e\cdot x$ est défini cet élément de $E$ est égal à $x$;

3. pour tout $x \in E$, il existe un $g \in \Gamma$ tel que $g\cdot x$ soit defini.

Par exemple si $h$: $\Gamma \rightarrow \Gamma'$ est un homomorphisme de groupoïdes qui envoie bijectivement les unités de $\Gamma$ sur celles de $\Gamma'$, on définit une telle loi de composition en posant

$$g\cdot x = h(g)\cdot x, \qquad g \in \Gamma, \qquad x \in \Gamma' = E$$

si le second membre est défini. On obtient une loi analogue en se limitant au

sous ensemble $E \subset \Gamma'$ des éléments $x$ qui ont une unité à droite donnée. Par ce procédé et en faisant $h = \delta$, on définit une opération partiellement définie de $\mathfrak{C}_a^1(\mathfrak{U}, \Phi)$ sur l'ensemble $C_a^2(\mathfrak{U}, \Phi)$ des 2-cochaînes fondamentales alternées.

PROPOSITION 6.3. *La loi de composition ainsi définie fait de* $\mathfrak{C}_a^1(\mathfrak{U}, \Phi)$ *un groupoïde d'opérateurs* (a) *sur l'ensemble* $C_a^2(\mathfrak{U}, \Phi)$ *des 2-cochaînes alternées fondamentales*; (b) *sur l'ensemble* $Z_a^2(\mathfrak{U}, \Phi)$ *des 2-cocycles alternés fondamentaux.*

*Démonstration* (a). [On fait, sans référence, un usage constant des relations (4.2), (4.3), (4.4), (5.1), (6.2).] Il faut montrer que si $(g_{ij}, \alpha_{ij}) \in \mathfrak{C}_a^1(\mathfrak{U}, \Phi)$ et $(\alpha_{ij}, \gamma_{ijk}, \epsilon_{ij}) \in C_a^2(\mathfrak{U}, \Phi)$ on a encore

$$\delta(g, \alpha)\cdot(\alpha, \gamma, \epsilon) \in C_a^2(\mathfrak{U}, \Phi).$$

D'après la proposition 5.3, deux relations seulement sont à vérifier:

$$(\delta_\alpha g)_{kij}\cdot\gamma_{kij} = (g\alpha)_{ki}[(\delta_\alpha g)_{ijk}\cdot\gamma_{ijk}],$$

$$(\delta_\alpha g)_{ikj}\cdot\gamma_{ikj} = \gamma_{ijk}^{-1}\cdot(\delta_\alpha g)_{ijk}^{-1}.$$

Pour les établir on doit faire usage de la condition (4.2) qui, pour une 2-cochaîne fondamentale $(\alpha_{ij}, \gamma_{ijk}, \epsilon_{ij})$, se réduit à

$$\alpha_{ij}\alpha_{jk}\alpha_{ki} = \rho(\gamma_{ijk}), \quad \text{d'où} \quad \alpha_{ij}\alpha_{jk}\alpha_{ki}(x) = \gamma_{ijk}\, x\, \gamma_{ijk}^{-1}.$$

En ce qui concerne la première, on a

$$\begin{aligned}(\delta_\alpha g)_{kij}\cdot\gamma_{kij} &= g_{ki}\cdot\alpha_{ki}(g_{ij})\cdot\alpha_{ki}\alpha_{ij}(g_{jk})\cdot(\gamma_{kij}\cdot g_{ki}\cdot\gamma_{kji})\cdot\gamma_{kij}g_{ki}^{-1}\\ &= g_{ki}\cdot\{\alpha_{ki}[g_{ij}\cdot\alpha_{ij}(g_{jk})\cdot\alpha_{ij}\alpha_{jk}(g_{ki})\cdot\gamma_{ijk}]\}\cdot g_{ki}^{-1}\\ &= (g\alpha)_{ki}[(\delta_\alpha g)_{ijk}\cdot\gamma_{ijk}].\end{aligned}$$

En ce qui concerne la seconde, on va montrer que

$$[(\delta_\alpha g)_{ikj}\cdot\gamma_{ikj}]^{-1} = (\delta_\alpha g)_{ijk}\cdot\gamma_{ijk}.$$

Cela résulte de

$$\begin{aligned}[(\delta_\alpha g)_{ikj}\cdot\gamma_{ikj}]^{-1} &= \gamma_{ijk}\cdot[g_{ik}\cdot\alpha_{ik}(g_{kj})\cdot\alpha_{ik}\alpha_{kj}(g_{ji})]^{-1}\\ &= \gamma_{ijk}\cdot\alpha_{ik}\alpha_{kj}\alpha_{ji}(g_{ij})\cdot\alpha_{ik}\alpha_{kj}(g_{jk})\cdot\alpha_{ik}(g_{ki})\\ &= \gamma_{ijk}\cdot\alpha_{ik}\alpha_{kj}\alpha_{ji}[g_{ij}\cdot\alpha_{ij}(g_{jk})\cdot\alpha_{ij}\alpha_{jk}(g_{ki})]\\ &= \gamma_{ijk}\cdot\gamma_{ikj}\cdot(\delta_\alpha g)_{ijk}\cdot\gamma_{ijk} = (\delta_\alpha g)_{ijk}\cdot\gamma_{ijk}.\end{aligned}$$

*Démonstration* (b). Il s'agit de montrer que si $(g_{ij}, \alpha_{ij}) \in \mathfrak{C}_a^1(\mathfrak{U}, \Phi)$ et $(\alpha_{ij}, \gamma_{ijk}, \epsilon_{ij}) \in Z_a^2(\mathfrak{U}, \Phi)$—c'est-à-dire si (6.2) est vérifié—on a encore

$$(\bar\alpha_{ij}, \bar\gamma_{ijk}, \epsilon_{ij}) = \delta(g, \alpha)\cdot(\alpha, \gamma, \epsilon) \in Z_a^2(\mathfrak{U}, \Phi),$$

$$\bar\alpha_{ij} = (g\alpha)_{ij}, \quad \bar\gamma_{ijk} = (\delta_\alpha g)_{ijk}\cdot\gamma_{ijk}.$$

On doit vérifier que l'on a dans $U_{ijkl}$

$$\bar{\gamma}_{ijk} = \bar{\gamma}_{ijl}\cdot\bar{\alpha}_{il}(\bar{\gamma}_{ljk})\cdot\bar{\gamma}_{ilk}.$$

Le second membre vaut successivement

$$(\delta_a g)_{ijl}\cdot\gamma_{ijl}\cdot[\bar{\alpha}_{il}((\delta_a g)_{ljk})\cdot\bar{\alpha}_{il}(\gamma_{ljk})]\cdot(\delta_a g)_{ilk}\cdot\gamma_{ilk}$$
$$= \{(\delta_a g)_{ijl}\cdot[\gamma_{ijl}\cdot\bar{\alpha}_{il}((\delta_a g)_{ljk})\gamma_{ijl}^{-1}][\gamma_{ijl}\cdot\bar{\alpha}_{il}(\gamma_{ljk})\cdot(\delta_a g)_{ilk}\cdot\alpha_{il}(\gamma_{ljk}^{-1})\cdot\gamma_{ijl}^{-1}]\}$$
$$\{\gamma_{ijl}\cdot\alpha_{il}(\gamma_{ljk})\cdot\gamma_{ilk}\}.$$

La seconde accollade n'est autre que $\gamma_{ijk}$ et on va vérifier que la première vaut $(\delta_a g)_{ijk}$. En effet elle est égale successivement à

$$(\delta_a g)_{ijl}\cdot\{\alpha_{ij}\alpha_{jl}\alpha_{li}[g_{il}\cdot\alpha_{il}((\delta_a g)_{ljk})g_{li}^{il}]\}\cdot[\ldots]$$
$$= g_{ij}^{i}\cdot g_{jl}^{ij}\cdot g_{li}^{ijl}\cdot g_{il}^{ijli}\cdot g_{ij}^{ijl}\cdot g_{jk}^{ij}\cdot g_{kl}^{ijk}\cdot g_{li}^{ijl}\cdot[\ldots]$$
$$= [g_{ij}^{i}\cdot g_{jk}^{ij}\cdot g_{kl}^{ijk}\cdot g_{li}^{ijl}]\cdot[\ldots]$$
$$= [\ldots]\cdot\alpha_{ij}\alpha_{jl}\alpha_{li}[g_{il}\cdot\alpha_{il}(\gamma_{ljk})\cdot g_{li}^{il}\cdot g_{il}^{l}\cdot g_{lk}^{il}\cdot g_{ki}^{ilk}\cdot\alpha_{il}(\gamma_{ljk}^{-1})]$$
$$= [\ldots]\cdot g_{il}^{ijli}\cdot\alpha_{ij}\alpha_{jl}(\gamma_{ljk}\cdot g_{lk}^{l}\cdot g_{kl}^{lk}\cdot\gamma_{ljk}^{-1})$$
$$= [\ldots]\cdot g_{il}^{ijli}\cdot\alpha_{ij}\alpha_{jl}\alpha_{lj}\alpha_{jk}\alpha_{kl}(g_{lk}^{l}\cdot g_{kl}^{lk})$$
$$= g_{ij}^{i}\cdot g_{jk}^{ij}\cdot(g_{kl}^{ijk}\cdot g_{li}^{ijl}\cdot g_{il}^{ijli}\cdot g_{lk}^{ijkl})\cdot g_{ki}^{ijk}$$
$$= g_{ij}^{i}\cdot g_{jk}^{ij}\cdot g_{ki}^{ijk}.$$

C.q.f.d.

**7. Classes de 2-cohomologie.** En coefficients abéliens, les classes de 2-cohomologie s'obtiennent en ajoutant à un 2-cocycle $z$ le cobord des 1-cochaînes $y$. Cette opération correspond ici à la multiplication à gauche de $z \in Z_a^2$ par les $\delta y$, $y \in \mathfrak{C}_a^1$:

$$z \to \delta y\cdot z.$$

Parmi les 2-cocycles alternés fondamentaux figurent les cobords des éléments $y$ de $C_a^1$(cochaînes alternés fondamentales) et, en coefficients abéliens, un cobord $\delta y$ n'est pas altéré en ajoutant à $y$ le cobord d'une 0-cochaîne $x \in C^0(\mathfrak{U}, \mathfrak{G})$. Il n'en va plus de même dans l'analogue non abélien. En effet si $y = (g_{ij}, \epsilon_{ij})$, $x = (h_i)$, "ajouter à $y$ le cobord de $x$"* se traduit par le passage de

$$(7.1)\qquad y = (g_{ij}, \epsilon_{ij}) \quad \text{à} \quad \bar{y} = x \,\square\, y = (h_i g_{ij} h_j^{-1}, \epsilon_{ij}).$$

L'opération $\square$ ainsi définie équivaut à faire opérer le groupe $C^0(\mathfrak{U}, \mathfrak{G})$ à gauche sur $C^1(\mathfrak{U}, \mathfrak{G})$. Simultanément les cobords passent de

$$\delta y = (\alpha_{ij}, \gamma_{ijk}, \epsilon_{ij}) \quad \text{à} \quad \delta\bar{y} = (\eta_i\alpha_{ij}\eta_j^{-1}, \eta_i(\gamma_{ijk}), \epsilon_{ij}),$$
$$\alpha_{ij} = \rho(g_{ij}), \quad \gamma_{ijk} = g_{ij}g_{jk}g_{ki}, \quad \eta_i = \rho(h_i).$$

*Voir à ce sujet (**1**; **2**; **8**; **10**; **11**).

Ceci conduit à faire opérer le groupe $C^0(\mathfrak{U}, \mathfrak{A})$ (et donc $C^0(\mathfrak{U}, \mathfrak{G})$) sur $C_a^2(\mathfrak{U}, \Phi)$ au moyen de

(7.2) $z = (\alpha_{ij}, \gamma_{ijk}, \epsilon_{ij}) \to \xi * z = (\eta_i \alpha_{ij} \eta_j^{-1}, \eta_i(\gamma_{ijk}), \epsilon_{ij}), \epsilon = (\eta_i) \in C^0(\mathfrak{U}, \mathfrak{A})$.

On doit évidemment vérifier que si $z$ est alternée il en est de même de $\xi * z$ et que $C^0(\mathfrak{U}, \mathfrak{A})$ opère donc bien sur $C_a^2(\mathfrak{U}, \Phi)$. De même $C^0(\mathfrak{U}, \mathfrak{G})$ opère aussi sur $C_a^2(\mathfrak{U}, \Phi)$ et on posera

$$x * z = \rho x * z, \qquad x \in C^0(\mathfrak{U}, \mathfrak{G}).$$

Ce qui précède démontre la propriété suivante:

Proposition 7.1. *Si $x \in C^0(\mathfrak{U}, \mathfrak{G})$ et $y \in C^1(\mathfrak{U}, \Phi)$, on a*

$$\delta(x \square y) = x * \delta y.$$

On établit encore:

Proposition 7.2. *Si $\xi$ appartient à $C^0(\mathfrak{U}, \mathfrak{A})$ et $z$ à $Z_a^2(\mathfrak{U}, \Phi)$ il en est de même de $\xi * z$.*

Autrement dit: $C^0(\mathfrak{U}, \mathfrak{A})$ *opère sur les 2-cocycles.*

*Démonstration.* Soient $z = (\alpha_{ij}, \gamma_{ijk}) \in Z_a^2$ et $\xi = (\eta_i) \in C^0(\mathfrak{U}, \mathfrak{A})$. D'où $\xi * z = (\alpha_{ij}', \gamma_{ijk}')$, $\alpha_{ij}' = \eta_i \alpha_{ij} \eta_j^{-1}$, $\gamma_{ijk}' = \eta_i(\gamma_{ijk})$. On doit vérifier que (6.2) entraîne

$$\gamma'_{ijk} = \gamma'_{iji} \cdot \alpha'_{ii}(\gamma'_{ijk}) \cdot \gamma'_{iik}.$$

Cela résulte de ce que le second membre vaut

$$\eta_i(\gamma_{iji}) \cdot \eta_i \alpha_{ii} \eta_i^{-1}(\eta_i(\gamma_{ijk})) \cdot \eta_i(\gamma_{iik}) = \eta_i(\gamma_{iji} \cdot \alpha_{ii}(\gamma_{ijk}) \cdot \gamma_{iik}) = \eta_i(\gamma_{ijk}) = \gamma'_{ijk}.$$

C.q.f.d.

L'opération $\square$ de $C^0(\mathfrak{U}, \mathfrak{G})$ sur $C_a^1(\mathfrak{U}, \mathfrak{G})$[1] s'étend à $\mathfrak{C}_a^1(\mathfrak{U}, \Phi)$ en posant pour $x = (h_i)$, $y = (g_{ij}, \alpha_{ij})$:

$$x \square y = (g'_{ij}, \alpha_{ij}) = (h_i g_{ij} \alpha_{ij}(h_j^{-1}), \alpha_{ij}),$$

ce qui est encore alterné vu que

$$[h_i g_{ij} \alpha_{ij}(h_j^{-1})]^{-1} = \alpha_{ij}(h_j) \cdot g_{ij}^{-1} \cdot h_i^{-1} = \alpha_{ij}[h_j \cdot g_{ji} \cdot \alpha_{ji}(h_i^{-1})].$$

Il existe une autre manière de faire opérer $C^0(\mathfrak{U}, \mathfrak{G})$ et plus généralement $C^0(\mathfrak{U}, \mathfrak{A})$ sur $\mathfrak{C}_a^1(\mathfrak{U}, \Phi)$, à savoir

(7.3) $$(x, y) \to x * y = (\eta_i(g_{ij}), \eta_i \alpha_{ij} \eta_j^{-1}) = (\bar{g}_{ij}, \bar{\alpha}_{ij})$$

ce qui est encore alterné en vertu de

$$\bar{g}_{ij}^{-1} = \eta_i(g_{ij}^{-1}) = \eta_i \alpha_{ij}(g_{ji}) = \eta_i \alpha_{ij} \eta_j^{-1}[\eta_j(g_{ji})] = \bar{\alpha}_{ij}(\bar{g}_{ji}).$$

[1] Cfr. la définition 4.3.

PROPOSITION 7.3. *Si* $\xi, y, z$ *sont des éléments respectivement de* $C^0(\mathfrak{U}, \mathfrak{A})$, $\mathfrak{C}_a^1(\mathfrak{U}, \Phi)$, $C_a^2(\mathfrak{U}, \Phi)$ *et si l'un des deux termes* $\xi*(\delta y\cdot z)$, $\delta(\xi*y)\cdot(\xi*z)$ *est défini, l'autre l'est et ils sont égaux*:

$$\xi*(\delta y\cdot z) = \delta(\xi*y)\cdot(\xi*z).$$

*Démonstration*. Soient $\xi = (\eta_i)$, $y = (g_{ij}, \alpha_{ij})$, $z = (\alpha_{ij}, \gamma_{ijk})$. On a (cfr. (2.2), (4.4), (7.2), (7.3))

$$\begin{aligned}
\xi*(\delta y\cdot z) &= (\eta_i\alpha'_{ij}\eta_j^{-1}, \eta_i[(\delta_a g)_{ijk}\cdot\gamma_{ijk}]), \quad \alpha'_{ij} = \rho(g_{ij})\circ\alpha_{ij};\\
\delta(\xi*y) &= \delta(\eta_i(g_{ij}), \eta_i\alpha_{ij}\eta_j^{-1})\\
&= (\beta_{ij}, \eta_i(g_{ij})\cdot\eta_i\alpha_{ij}\eta_j^{-1}\eta_j(g_{jk})\cdot\eta_i\alpha_{ij}\alpha_{jk}\eta_k^{-1}\eta_k(g_{ki}), \eta_i\alpha_{ij}\eta_j^{-1})\\
&= (\beta_{ij}, \eta_i[(\delta_a g)_{ijk}], \eta_i\alpha_{ij}\eta_j^{-1}) \qquad\qquad (\beta_{ij} = \rho[\eta_i(g_{ij})]\circ\eta_i\alpha_{ij}\eta_j^{-1})\\
&= (\eta_i\alpha'_{ij}\eta_j^{-1}, \eta_i[(\delta_a g)_{ikj}], \eta_i\alpha_{ij}\eta_j^{-1});\\
\xi*z &= (\eta_i\alpha_{ij}\eta_j^{-1}, \eta_i(\gamma_{ijk}))
\end{aligned}$$

La proposition s'en déduit immédiatement. C.q.f.d.

D'après les propositions 6.3 et 7.2, pour tout 2-cocycle $z \in Z_a^2(\mathfrak{U}, \Phi)$, l'élément $\xi*(\delta y\cdot z)$ (supposé défini) est encore un cocycle. Deux cocycles $z$ et $z'$ sont dits ***équivalents*** si $z'$ est de la forme

$$z' = \xi*(\delta y\cdot z).$$

Cette relation est manifestement réflexive, et on vérifie sans peine qu'elle est transitive et symétrique grace au diagramme et à l'égalité suivants

$$\begin{tikzcd}
 & \delta y\cdot z \arrow[d] & z \arrow[l] \arrow[d] \\
\delta y'\cdot z' \arrow[d] & z' = \xi*(\delta y\cdot z) \arrow[l] \arrow[d] & \xi*z \arrow[l] \arrow[d] \\
z'' = \xi'*(\delta y'\cdot z') & \xi'*z' \arrow[l] & (\xi'\cdot\xi)*z \arrow[l]
\end{tikzcd}$$

$$z' = \xi*(\delta y\cdot z) = \delta(\xi*y)\cdot(\xi*z) \leftrightarrows z = \xi^{-1}*[\delta(\xi*y)^{-1}\cdot z'].$$

N.B. Dans le diagramme les flèches horizontales (resp. verticales) indiquent une multiplication par un 2-cobord (resp. le résultat d'un opérateur $z \to \xi*z$).

*Définition* 7.4. On appelle *classe de cohomologie de dimension deux de* $\mathfrak{U}$ *à coefficients dans* $\Phi$ toute classe de cocycles équivalents au sens précédent. La classe *nulle* est celle des éléments $z = \xi*\delta y$, $\xi \in C^0(\mathfrak{U}, \mathfrak{A})$, $y \in C_a^1(\mathfrak{U}, \Phi)$. En outre on distinguera encore les classes qui contiennent un élément de la forme $(\alpha_{ij}, \epsilon_{ijk}, \epsilon_{ij})$ où $\epsilon$ représente une section unité (dans ce cas $\alpha_{ij}$ est automatiquement un cocycle de dimension 1 à coefficients dans $\mathfrak{A}$); ces classes seront appelées *neutres*.

*Cfr. la définition 4.3.

Il est clair que ces classes coïncident avec les classes usuelles dans le cas abélien. Cette définition se justifiera encore par la suite, voir notamment la proposition 8.1 et la définition de l'opérateur cobord (9.4) de la suite exacte. L'ensemble des classes sera noté $H^2(\mathfrak{U}, \Phi)$.

**8. Passage à la limite inductive.** Soient $\mathfrak{U} = (U_i)_{i \in I}$, $\mathfrak{B} = (V)_{r \in R}$ deux recouvrements couverts de $X$. Si $\mathfrak{B}$ est plus fin que $\mathfrak{U}$, il existe une appilcation, dite admissible, $\phi$: $R \to I$ telle que $V_r \subset U_{\phi r}$ ce qui induit des applications que nous noterons encore $\phi$:

$$\phi\colon C^0(\mathfrak{U}, \mathfrak{G}) \to C^0(\mathfrak{B}, \mathfrak{G}),\ C^0(\mathfrak{U}, \mathfrak{A}) \to C^0(\mathfrak{B}, \mathfrak{A}),$$

$$\phi\colon \mathfrak{C}^1_a(\mathfrak{U}, \Phi) \to \mathfrak{C}^1_a(\mathfrak{B}, \Phi),$$

$$\phi\colon \mathfrak{C}^2_a(\mathfrak{U}, \Phi) \to \mathfrak{C}^2_a(\mathfrak{B}, \Phi),\ \text{etc.}\ldots$$

qui sont compatibles avec les opérations considérées plus haut et qui transforment un cocycle en un cocycle:

$$\begin{aligned} \phi(x \square y) &= \phi(x) \square \phi(y),\\ \phi(x * y) &= \phi(x) * \phi(y),\\ \phi(\delta y) &= \delta\phi(y),\\ \phi(x * z) &= \phi(x) * \phi(z). \end{aligned}$$

Dès lors $\phi$ induit une application

$$i(\mathfrak{B}, \mathfrak{U})\colon H^2(\mathfrak{U}, \Phi) \to H^2(\mathfrak{B}, \Phi).$$

PROPOSITION 8.1. *Cette application est indépendante de l'application admissible $\phi$: $R \to I$ choisie. De façon précise si $\phi' = R \to I$ en est une autre et si $z \in Z_a^2(\mathfrak{U}, \Phi)$ les cocycles $\phi(z)$ et $\phi'(z)$ relatifs au recouvrement plus fin $\mathfrak{B}$ sont équivalents.*

*Démonstration.* Soit $z = (\alpha_{ij}, \gamma_{ijk}, \epsilon_{ij}) \in Z_a^2\ (\mathfrak{U}, \Phi)$ et pour trois indices $r, s, t$ du recouvrement $\mathfrak{B}$, posons

$$\phi(r) = i, \qquad \phi(s) = j, \qquad \phi(t) = k,$$
$$\phi'(r) = i', \qquad \phi'(s) = j', \qquad \phi'(t) = k'.$$

Les cocycles images de $z$ par $\phi$ et $\phi'$ s'obtiennent par des restrictions appropriées que nous écrivons brièvement

$$\phi(z) = (\alpha_{rs}, \gamma_{rst}, \epsilon_{rs}) = (\alpha_{ij}, \gamma_{ijk}, \epsilon_{ij})$$
$$\phi'(z) = (\alpha'_{rs}, \gamma'_{rst}, \epsilon'_{rs}) = (\alpha_{i'j'}, \gamma_{i'j'k'}, \epsilon_{i'j'}).$$

La proposition sera établie si nous exhibons $\xi = (\eta_r) \in C^0(\mathfrak{B}, \mathfrak{A})$, $y = (y_{rs}, \alpha_{rs}) \in \mathfrak{C}_a^1(\mathfrak{B}, \Phi)$ tels que

$$\xi * \phi'(z) = \delta y \cdot \phi(z). \tag{8.1}$$

La solution est, nous allons le voir, fournie par

$$\eta_r = \alpha_{ii'}|V_r$$
$$y_{rs} = \gamma_{ii'j'}\gamma_{ij'j}|V_{rs}.$$

La première chose à vérifier est l'alternation de $(y_{rs}, \alpha_{rs})$, ce qui résulte de

$$\gamma_{ii'j'} = \gamma_{ii'j}\cdot\alpha_{ij}(\gamma_{ji'j'})\cdot\gamma_{ijj'},$$
$$y_{rs} = \gamma_{ii'j'}\gamma_{ij'j} = \gamma_{ii'j}\cdot\alpha_{ij}(\gamma_{ji'j'}) = \alpha_{ij}(\gamma_{jii'}\cdot\gamma_{ji'j'}),$$
$$y_{rs}^{-1} = \alpha_{ij}(\gamma_{jj'i'}\cdot\gamma_{ji'i}) = \alpha_{rs}(y_{sr}).$$

Le premier membre de (8.1) est

$$(\alpha_{ii'}\alpha_{i'j'}\alpha_{j'j}, \alpha_{ii'}(\gamma_{i'j'k'}), \epsilon_{i'j'}) = (\eta_r\alpha'_{rs}\eta_s^{-1}, \eta_r(\gamma'_{rst}), \epsilon_{rs}) \tag{8.2}$$

et on a, par des applications répétées des relations (4.2), (4.4), (6.2):

$$\begin{aligned}
\eta_r(\gamma'_{rst}) &= \alpha_{ii'}(\gamma_{i'j'k'}) = \alpha_{ii'}(\gamma_{i'j'i}\cdot\alpha_{i'i}(\gamma_{ij'k'})\cdot\gamma_{i'ik'})\\
&= \gamma_{ii'j'}\cdot\gamma_{ij'k'}\cdot\gamma_{ik'i'} = (\gamma_{ii'j'}\cdot\gamma_{ij'j})\cdot\alpha_{ij}(\gamma_{jj'k'})\cdot\gamma_{ijk'}\cdot\gamma_{ik'i'}\\
&= y_{rs}\cdot\alpha_{ij}(\gamma_{jj'k'}\cdot\gamma_{jk'i})\cdot\gamma_{ik'i'}\\
&= y_{rs}\cdot\alpha_{ij}[\gamma_{jj'k'}\cdot\gamma_{jk'k}\cdot\alpha_{jk}(\gamma_{kk'i})\cdot\gamma_{jki}]\cdot\gamma_{ik'i'}\\
&= y_{rs}\cdot\alpha_{rs}(y_{st})\cdot\alpha_{ij}\alpha_{jk}(\gamma_{kk'i}\cdot\gamma_{kij})\cdot\gamma_{ik'i'}\\
&= y_{rs}\cdot\alpha_{rs}(y_{st})\cdot\alpha_{ij}\alpha_{jk}[\gamma_{kk'i'}\cdot\alpha_{ki'}(\gamma_{i'k'i})\cdot\gamma_{ki'i}]\cdot\gamma_{ijk}\cdot\gamma_{ik'i'}\\
&= y_{rs}\cdot\alpha_{rs}(y_{st})\cdot\alpha_{ij}\alpha_{jk}[(\gamma_{kk'i'}\gamma_{ki'i})\cdot\gamma_{kii'}\cdot\alpha_{ki'}(\gamma_{i'k'i})\gamma_{ki'i}]\cdot\gamma_{ijk}\cdot\gamma_{ik'i'}\\
&= y_{rs}\cdot\alpha_{rs}(y_{st})\cdot\alpha_{rs}\alpha_{st}(y_{tr})\cdot\alpha_{ij}\alpha_{jk}\alpha_{ki}\alpha_{ii'}\alpha_{i'k}\alpha_{ki'}(\gamma_{i'k'i})\cdot\gamma_{ijk}\cdot\gamma_{ik'i'}\\
&= (\delta_\alpha y)_{rst}\cdot\gamma_{ijk}\cdot\gamma_{ii'k'}\cdot\gamma_{ijk}^{-1}\cdot\gamma_{ijk}\cdot\gamma_{ik'i'}\\
&= (\delta_\alpha y)_{rst}\cdot\gamma_{ijk} = (\delta_\alpha y)_{rst}\cdot\gamma_{rst}.
\end{aligned}$$

Ceci montre que les termes centraux à trois indices des deux membres de (8.1) sont égaux. Il reste à prouver que

$$\eta_r\alpha'_{rs}\eta_s^{-1} = \rho(y_{rs})\circ\alpha_{rs}$$

ce qui résulte de

$$\begin{aligned}
\eta_r\alpha'_{rs}\eta_s^{-1} &= \alpha_{ii'}\alpha_{i'j'}\alpha_{j'j},\\
\rho(y_{rs})\circ\alpha_{rs} &= \rho(\gamma_{ii'j'}\cdot\gamma_{ij'j})\circ\alpha_{ij}\\
&= \alpha_{ii'}\alpha_{i'j'}\alpha_{j'i}\alpha_{ij'}\alpha_{j'j}\alpha_{ji}\alpha_{ij} = \alpha_{ii'}\alpha_{i'j'}\alpha_{j'j}.
\end{aligned}$$

Ceci achève la démonstration de la proposition.

*Définition* 8.1. La limite inductive des ensembles $H^2(\mathfrak{U}, \Phi)$ par rapport aux $i(\mathfrak{B}, \mathfrak{U})$ est appelée ***ensemble de cohomologie de dimension deux de l'espace*** $X$ **à** ***coefficients dans*** $\Phi$. Cet ensemble est noté

$$H^2(X, \Phi) = \varinjlim H^2(\mathfrak{U}, \Phi);$$

il possède: (a) un élément *nul* $\mathfrak{e}_2$ image des classes nulles des termes de la limite; (b) un sous ensemble $\mathrm{E}^2$ d'éléments *neutres* obtenu de façon analogue (*partie neutre*).

**9. Suite exacte.** Soit une suite exacte

$$e \to \mathfrak{N} \xrightarrow{i} \mathfrak{G} \xrightarrow{j} \mathfrak{H} \to e \tag{9.1}$$

de faisceaux de groupes (non abéliens) sur l'espace $X$. Il est bien connu qu'elle engendre la suite exacte de cohomologie (voir (**2**; **9**; **11**))

$$e \to H^0(X, \mathfrak{N}) \xrightarrow{i^0} H^0(X, \mathfrak{G}) \xrightarrow{j^0} H^0(X, \mathfrak{H}) \xrightarrow{\delta^0} H^1(X, \mathfrak{N}) \xrightarrow{i^1} H^1(X, \mathfrak{G}) \xrightarrow{j^1} H^1(X, \mathfrak{H}) \tag{9.2}$$

et que celle-ci peut d'ailleurs s'immerger dans une suite, plus précise, de groupoïdes à condition de remplacer le foncteur $H$ par $\mathfrak{H}$ ou $\mathfrak{H}_{\mathbf{Z}}$ (voir (**4**; **5**; **6**)).

[Rappelons que pour tout faisceau de groupes $\mathfrak{G}$ sur l'espace $X$, on designe par $H^0(X, \mathfrak{G})$ le groupe des sections globales $s$: $X \to \mathfrak{G}$. Pour un recouvrement ouvert $\mathfrak{U} = (U_i)_{i \in I}$ de $X$, une 1-cochaîne fondamentale $g_{ij}$ est appelée 1-cocycle si $g_{ij} = g_{ik}g_{kj}$. Deux 1-cocycles $g_{ij}$, $g_{ij}'$ sont dits cohomologues s'il existe une 0-cochaîne $h_i$: $U_i \to \mathfrak{G}$ telle que $g_{ij}' = h_i g_{ij} h_j^{-1}$. Les classes de cocycles cohomologues forment un ensemble $H^1(\mathfrak{U}, \mathfrak{G})$ de limite inductive $H^1(X, \mathfrak{G})$. La classe des 1-cobords, c'est-à-dire des 1-cocycles $g_{ij} = h_i h_j^{-1}$ est dite neutre et induit un élément neutre $\mathfrak{e}^1 \in H^1(X, \mathfrak{G})$.

L'exactitude de (9.2) est l'exactitude usuelle pour les groupes jusqu'au terme $H^0(X, \mathfrak{H})$; ensuite elle signifie que l'image d'une application coïncide avec l'image inverse de l'élément neutre par la suivante.

On notera que le foncteur $H$ utilise en dimensions 0 et 1 une notion moins précise de "système de coefficients," c'est-à-dire seul le faisceau $\mathfrak{G}$ intervient, le faisceau d'automorphismes $\mathfrak{A}$ et l'homomorphisme $p$ ne jouant aucun rôle. Voir la remarque 2.1.]

Nous allons considérer un système $\Phi = (\mathfrak{G}, \rho, \mathfrak{A})$ tel que les éléments de $\mathfrak{A}$ laissent invariant le sous-faisceau $\mathfrak{N}$ de $\mathfrak{G}$. Il en résulte alors un système $\Phi' = (\mathfrak{N}, \rho', \mathfrak{A})$ où $\rho' = \rho o i$: $\mathfrak{N} \to \mathfrak{A}$. Ces systèmes permettent de définir $H^2(X, \Phi')$ et $H^2(X, \Phi)$, ainsi qu'une application

$$i^2: H^2(X, \Phi') \to X^2(X, \Phi) \tag{9.3}$$

qui transforme une classe neutre (resp. nulle) en classe neutre (resp. nulle). On définit ensuite une opération

$$\delta^1: H^1(X, \mathfrak{H}) \to H^2(X, \Phi') \tag{9.4}$$

à condition que $X$ soit paracompact. A cet effet on représente un élément de $H^1(X, \mathfrak{H})$ par un cocycle $h_{ij} \in Z_a^1(\mathfrak{U}, \mathfrak{H})$ pour un recouvrement $\mathfrak{U}$ suffisam-

ment fin pour que $h_{ij}$ soit l'image d'un $g_{ij} \in C_a^1(\mathfrak{U}, \mathfrak{G})$[1]. Soient alors $\alpha_{ij} = \rho(g_{ij})$ et $\nu_{ijk} = (\delta g)_{ijk}$ le couple $(\alpha_{ij}, \nu_{ijk})$ est un élément de $Z_a^2(\mathfrak{U}, \Phi')$; en remplaçant $g_{ij}$ par un autre élément $g_{ij}'$ se projetant sur $h_{ij}$ et de même en faisant varier $h_{ij}$ dans sa classe de cohomologie le couple $(\alpha_{ij}, \nu_{ijk})$ ne sort pas d'une classe de cohomologie de $H^2(\mathfrak{U}, \Phi')$ dont l'image dans $H^2(X, \Phi')$ est elle-même indépendante de $\mathfrak{U}$. D'où l'application (9.4). Ces affirmations se démontrent aisément en considérant la diagramme (9.5) ci-dessous et en examinant ce qui se passe en *remplaçant* modifiant $g_{ij}$ dans l'image inverse de $h_{ij}$, puis $h_{ij}$ dans sa classe de cohomologie. Ce mécanisme est l'une des justifications de la definition 7.4 de classe de 2-cohomologie.

PROPOSITION 9.1. L'espace $x$ étant paracompact, *pour qu'un élément de $H^1(X, \mathfrak{H})$ soit dans l'image de $j^1$, il faut et il suffit que son image par $\delta^1$ soit une classe neutre de $H^2(X, \Phi')$.*

*Démonstration.* Au niveau d'un recouvrement $\mathfrak{U}$ convenable la définition de l'opérateur $\delta^1$ correspond au diagramme suivant (voir (4.5), (4.6))

$$\begin{array}{ccccc} & & g_{ij} & \xrightarrow{\quad j \quad} & h_{ij} \\ (9.5) & & \downarrow\delta & & \downarrow\delta \\ [\alpha_{ij}, \nu_{ijk} = (\delta g)_{ijk}] & \xrightarrow{i} & [\alpha_{ij} = \rho(g_{ij}), (\delta g)_{ijk}] & \xrightarrow{j} & [\alpha_{ij}, \epsilon_{ijk}] \end{array}$$

dans le quel $i$ et $j$ representent les applications induites par les homomorphismes de même nom dans (9.1). Il indique comment on passe du 1-cocycle $h_{ij}$ au 2-cocycle $z = (\alpha_{ij}, \nu_{ijk})$. Il est facile de voir que la condition nécessaire et suffisante pour que la classe de $z$ soit neutre et qu'il existe $y \in \mathfrak{C}_a^1(\mathfrak{U}, \Phi')$ de la forme $y = (\eta_{ij}, \alpha_{ij})$ tel que $\delta y \cdot z$ soit de la forme $(\alpha_{ij}', \epsilon_{ijk})$. Or $\delta y \cdot z = \delta(g_{ij}')$ avec $g_{ij}' = \eta_{ij} g_{ij}$; la condition est donc équivalente à l'existence de $(g_{ij}') \in \mathfrak{C}_a^1(\mathfrak{U}, \mathfrak{G})$ tel que

$$j(g_{ij}') = j(g_{ij}), \qquad (\delta g')_{ijk} = \epsilon_{ijk}. \qquad \text{C.q.f.d.}$$

PROPOSITION 9.2. *Pour qu'un élément de $H^2(X, \Phi')$ soit dans l'image de l'application $\delta^1$, il faut et il suffit que son image par $i^2$ soit la classe nulle de $H^2(X, \Phi)$.*

*Démonstration.* La condition est évidemment necessaire en vertu de la définition de $\delta^1$ (cf. le diagramme (9.5)). Soit $z = (\alpha_{ij}, \nu_{ijk})$ un élément de $Z_a^2(\mathfrak{U}, \Phi')$; sa classe est envoyée sur zéro par $i^2$ s'il existe $\xi = (\xi_i) \in C^0(\mathfrak{U}, \mathfrak{A})$, $y = (g_{ij}) \in C_a^1(\mathfrak{U}, \mathfrak{G})$ tel que $z = \xi * \delta y$ c'est-à-dire

$$\alpha_{ij} = \xi_i \rho(g_{ij}) \xi_j^{-1}, \quad \nu_{ijk} = \xi_i(g_{ij} g_{jk} g_{ki}).$$

[1] Un recouvrement suffisamment fin existe parcequ'on suppose $X$ paracompact. La démonstration est la même que dans le cas abélien. Voir par example [1, b] où le problème est examiné dans l'hypothèse où le faisceau $\mathfrak{N}$ est dans le centre de $\mathfrak{G}$.

Mais alors $z' = \xi^{-1} * z = \delta y$ est dans la classe $[z]$ de $z$ et $h_{ij} = j(g_{ij})$ est un cocycle de $Z_a^1(\mathfrak{U}, \mathfrak{H})$ dont la classe $[h]$ est envoyée sur $[z]$ par $\delta^1$. C.q.f.d.

*Suite exacte croisée.* Supposons qu'il existe associé à la suite exacte (9.1) un homomorphisme $k: \mathfrak{H} \to \mathfrak{G}$ tel que $j \circ k$ soit l'identité de $\mathfrak{H}$. Alors $\mathfrak{G}$ s'identifie à un produit direct croisé de $\mathfrak{N}$ et $\mathfrak{H}$ et nous dirons que la suite exacte (9.1) est *croisée.* En vertu de (2.2), $\rho(\mathfrak{N})$ est nécessairement distingué dans $\mathfrak{N}$ et nous pouvons former le quotient $\mathfrak{A}''$ et l'homomorphisme canonique:

$$\pi: \mathfrak{A} \to \mathfrak{A}'' = \mathfrak{A}/\rho(\mathfrak{N}).$$

Comme $\mathfrak{N}$ est invariant par $\mathfrak{A}$, on fait opérer $\mathfrak{A}$ sur $\mathfrak{H}$ en posant

$$\alpha(jg) = j(\alpha g), \qquad \alpha \in \mathfrak{A}, \qquad g \in \mathfrak{G}$$

et les éléments $\alpha = \rho(n)$, $n \in \mathfrak{N}$ opèrent trivialement vu que

$$\rho(n)(jg) = j[\rho(n)\cdot g] = j(ngn^{-1}) = jg.$$

Il s'ensuit que $\mathfrak{A}''$ opère sur $\mathfrak{H}$ de telle sorte que

$$(\pi\alpha)(jg) = j(\alpha g).$$

On va voir que si $\rho'' = \pi\rho k: \mathfrak{H} \to \mathfrak{A}''$,

$$\rho''(jg) = \pi(\rho(g)).$$

En effet $\rho''(jg) = \pi\rho kjg$ et, comme $kjg$ et $g$ ont même image $jg = h$ dans $\mathfrak{H}$, on a $kjg = g\cdot n$, $n \in \mathfrak{N}$. Dès lors

$$\rho''(jg) = \pi\rho(g\cdot h) = \pi[\rho(g)\cdot\rho(n)] = \pi[\rho(g)].$$

Il suit de là que

$$\rho''(h)(jg') = \pi\rho(g)\cdot(jg') = j[\rho(g)(g')] = j(gg'g^{-1}) = h\cdot jg'\cdot h^{-1},$$
$$\rho''[\pi\alpha(jg)] = \rho''[j\alpha(g)] = \pi\rho kj\alpha(g) = \pi\rho\alpha(g) = \pi[\alpha \circ \rho(g) \circ \alpha^{-1}]$$
$$= (\pi\alpha) \circ \pi\rho(g) \circ (\pi\alpha)^{-1} = (\pi\alpha) \circ \rho''(jg) \circ (\pi\alpha)^{-1}.$$

Ceci montre que l'homomorphisme $\rho'': \mathfrak{H} \to \mathfrak{A}''$ vérifie (2.1) et (2.2) et que $\Phi'' = (\mathfrak{H}, \rho'', \mathfrak{A}'')$ est un nouveau système de coefficients. Si $z = (\alpha_{ij}, \gamma_{ijk}) \in C_a^2(\mathfrak{U}, \Phi)$, on forme $z'' = j_* z = (\pi\alpha_{ij}, j\,\gamma_{ijk})$ et on a

$$\rho''(j\gamma_{ijk}) = \pi\rho(\gamma_{ijk}) = \pi(\alpha_{ij}\alpha_{jk}\alpha_{ki}) = \pi\alpha_{ij} \circ \pi\alpha_{jk} \circ \pi\alpha_{ki},$$

c'est-à-dire $z'' \in C_a^2(\mathfrak{U}, \Phi'')$. Ceci donne lieu à une application (compatible avec les parties neutres et les éléments nuls):

$$j^2: H^2(X, \Phi) \to H^2(X, \Phi'').$$

PROPOSITION 9.3. *Si la suite* (9.1) *est croisée, pour qu'un élément de* $H^2(X, \Phi)$ *soit dans l'image de* $i^2$, *il faut et il suffit que son image par* $j^2$ *soit neutre.*

*Démonstration.* Conservons les notations précédentes et supposons que $z \in Z_a^2(\mathfrak{U}, \Phi)$. Si $z$ provient de $Z_a^2(\mathfrak{U}, \Phi')$, on a $\gamma_{ijk} \in \mathfrak{N}$ et la partie "il faut" est

immédiate. Réciproquement, supposons que la classe de $z''$ soit neutre: il existe donc $\zeta'' = (\beta_{ij}'', \epsilon_{ijk}) \in Z_a^2(\mathfrak{U}, \Phi'')$, $y'' = (h_{ij}, \pi\alpha_{ij}) \in \mathfrak{C}_a^1(\mathfrak{U}, \Phi'')$ tel que $\zeta'' = \delta y'' \cdot z''$:

$$\beta_{ij}'' = \rho''(h_{ij}) \circ \pi\alpha_{ij}, \qquad 1_{ijk} = (\delta_{\pi a} h)_{ijk} \cdot j\gamma_{ijk}.$$

Soit $y = (g_{ij}, \alpha_{ij}) \in \mathfrak{C}_a^1(\mathfrak{U}, \Phi)$ tel que $jg_{ij} = h_{ij}$ et soit $\zeta = \delta y \cdot z$:

$$\zeta = (\beta_{ij}, (\delta_a g)_{ijk} \cdot \gamma_{ijk}), \qquad \beta_{ij} = \rho(g_{ij}) \circ \alpha_{ij}.$$

On a $j^*\zeta = \zeta''$ et $j[(\delta_a g)_{ijk} \cdot \gamma_{ijk}] = \epsilon_{ijk}$, ce qui montre que $\zeta$ provient d'un élément de $Z_a^1(\mathfrak{U}, \Phi')$. C.q.f.d.

Les résultats classiques en dimension zéro et un augmentés de ceux de ce paragraphe donnent lieu au

THÉORÈME 9.4. *Si l'espace $X$ est paracompact, à la suite exacte* (resp. exacte croisée) (9.1) *est associée la suite exacte de cohomologie*

$$\begin{aligned} e &\to H^0(X, \mathfrak{N}) \to H^0(X, \mathfrak{G}) \to H^0(X, \mathfrak{H}) \\ &\to H^1(X, \mathfrak{N}) \to H^1(X, \mathfrak{G}) \to H^1(X, \mathfrak{H}) \\ &\to H^2(X, \Phi') \to H^2(X, \Phi)[\to H^2(X, \Phi'')]. \end{aligned}$$

Pour comprendre le sense de l'*exactitude*, les termes de dimension deux doivent être munis d'un *sous-ensemble privilégié* qui est celui des classes neutres pour $H^2(X, \Phi')$ [et $H^2(X, \Phi'')$] et celui réduit à la classe nulle pour $H^2(X, \Phi)$. L'exactitude signifie que l'image d'une application coïncide avec l'image inverse du sous-ensemble privilégié par l'application suivante.

*Note ajoutée à la correction des épreuves.* Dans le cas où $\mathfrak{A} = \mathfrak{J}(\mathfrak{G})$, M. D. Puppe me fait remarquer que l'hypothèse d'une suite exacte croisée est superflue pour définir le dernier terme de la suite exacte du théorème (9.4). Cette hypothèse est même superflue dans tous les cas. En effet de la condition (2.2) et de l'invariance de $\mathfrak{N}$ par $\mathfrak{A}$ il résulte que $\rho(\mathfrak{N})$ est un sous-groupe invariant dans $\mathfrak{B}$ et que $\mathfrak{A}'' = \mathfrak{A}/\rho(\mathfrak{N})$ est donc un groupe. Or tout élément de $\rho(\mathfrak{N})$ opérant sur un élément $g \in \mathfrak{G}$ le transforme en un $g'$ congru à $g$ modulo $\mathfrak{N}$. Il suit de là que $\mathfrak{A}''$ opère sur $\mathfrak{H}$ par $\phi'': \mathfrak{A}'' \times \mathfrak{H} \to \mathfrak{H}$ de telle sorte que l'on ait le diagramme commutatif suivant.

$$\begin{array}{ccc} \mathfrak{A} \times \mathfrak{G} & \xrightarrow{\pi \times j} & \mathfrak{A}'' \times \mathfrak{H} \\ \phi \downarrow & & \downarrow \phi'' \\ \mathfrak{G} & \xrightarrow{\quad j \quad} & \mathfrak{H} \end{array}$$

Ensuite on définit $\rho''$ de manière à avoir le diagramme commutatif:

$$\begin{array}{ccccccccc} e & \longrightarrow & \mathfrak{N} & \longrightarrow & \mathfrak{G} & \xrightarrow{j} & \mathfrak{H} & \longrightarrow & e \\ & & \downarrow \rho & & \downarrow \rho & & \downarrow \rho'' & & \\ e & \longrightarrow & \rho(\mathfrak{N}) & \longrightarrow & \mathfrak{A} & \xrightarrow{\pi} & \mathfrak{A}'' & \longrightarrow & e \end{array}$$

Comme dans le texte on vérifie alors que $(\mathfrak{H}, \rho'', \mathfrak{A}'')$ est un système de coefficients (en observant par exemple qu'on n'utilise plus le fait que $k$ est un homomorphisme).

**10. Application aux espaces fibrés.** Soit

$$e \to N \to G \to H \to e$$

une suite exacte de groupes topologiques où $N$, fermé dans $G$ possède une section locale. Alors les jets locaux des applications continues d'un espace $X$ dans $N$, $G$ et $H$ définissent des faisceaux $\mathfrak{N}$, $\mathfrak{G}$ et $\mathfrak{H}$ sur $X$ donnant lieu à une suite exacte (9.1). On sait que les éléments de $H^1(X, \mathfrak{H})$ par exemple s'identifient aux classes d'espaces fibrés principaux localement triviaux de base $X$ et de groupe structural $H$. Le problème se pose de déterminer si un espace fibré $E_H \in H^1(X, \mathfrak{H})$ est l'image canonique d'un espace fibré $E_G \in H^1(X, \mathfrak{G})$. Le théorème 9.4 (ou la proposition (9.1)) a pour corollaire immédiat.

THÉORÈME 10.1. *L'espace fibré principal $E_H$ dont la base $X$ est supposée paracompacte est l'image d'un $E_G$ si et seulement si $\delta^1 E_H$ est une classe neutre de $H^2(X, \Phi')$.*

Des applications concrètes de ce résultat demanderaient que l'on puisse calculer effectivement $H^2(X, \Phi')$ au moins dans certains cas ou seulement que l'on ait des résultats donnant des conditions pour que $H^2(X, \Phi')$ se réduise à l'ensemble neutre $E^2(X, \Phi')$. Une étude de la thèse de J. Frenkel [2, c] et de celle de H. Grauert. (à paraître aux Math. Annal.) conduirait vraisemblablement à des résultats de ce genre (voir aussi la communication de H. Cartan au symposium de Mexico, août 1956).

*Remarque.* Il y aurait lieu de compléter le théorème 9.4 de manière à pouvoir déterminer quand deux éléments distincts d'un $H^2$ ont la même image dans le $H^2$ suivant. Ceci pourrait se faire en remplaçant le foncteur-ensemble $H^2$ par un foncteur-groupoïde $\mathfrak{H}^2$ mais il faudrait alors remplacer les foncteurs $\mathfrak{H}^1$ que j'ai introduits précédemment par des foncteurs encore plus compliqués. La question se pose de savoir si l'on peut définir des foncteurs $H^n$ et $\mathfrak{H}^n$ pour $n$ quelconque se réduisant aux foncteurs habituels dans le cas des coefficients abéliens et donnant lieu à des suites exactes. Des résultats fragmentaires semblent indiquer que la chose est possible mais, en l'absence d'applications, il convient de ne généraliser qu'avec quelque prudence. La chose pourrait être possible par une modification de la technique des résolutions d'un faisceau par des daisceaux fins mais les "théorèmes d'unicité" présentent des difficultés.

## COMPLÉMENTS

**11. Acyclicité du tétraèdre.** Les idées qui précèdent peuvent s'appliquer également à la cohomologie d'un complexe simplicial abstrait $K$ à coefficients dans un groupe $G$ muni d'un homomorphisme $\rho$ dans un groupe $A$ d'automorphisme et satisfaisant à des conditions analogues à (2.1) et (2.2).

Supposons $K$ constitué par un tétraèdre de sommets 1, 2, 3, 4 et $A = I(G)$, $\rho$ étant l'homomorphisme canonique.

PROPOSITION 11.1. *Sous les hypothèses précédentes, tout 2-cocycle alterné fondamental $z$ de $K$ est un cobord.*

Le cocycle $z$ est un système alterné $(\alpha_{ij}, \gamma_{ijk})$, tel que pour $i, j, k, l = 1, 2, 3, 4$ on ait

(11.1) $\alpha_{ij} \in I(G)$.

(11.2) $\alpha_{ij}\alpha_{jk}\alpha_{ki}$ opère comme $\gamma_{ijk}$ par automorphismes intérieurs.

(11.3) $\gamma_{ijk} = \gamma_{ijl}\cdot\alpha_{il}(\gamma_{ljk})\cdot\gamma_{ilk}$.

On doit trouver des $g_{ij}$ alternés tels que

$$\rho(g_{ij}) = \alpha_{ij}$$
$$\gamma_{ijk} = g_{ij}g_{jk}g_{ki}$$

Choisissons arbitrairement $g_{14}$, $g_{24}$ et $g_{34}$ opèrant comme $\alpha_{14}$, $\alpha_{24}$ et $\alpha_{34}$. Désignons par $g_{41}$, $g_{42}$, $g_{43}$ leurs inverses (opérant comme $\alpha_{41}$, $\alpha_{42}$, $\alpha_{43}$). Les relations

$$\gamma_{124} = g_{12}g_{24}g_{41}$$
$$\gamma_{234} = g_{23}g_{34}g_{42}$$
$$\gamma_{314} = g_{31}g_{14}g_{43}$$

permettent de calculer $g_{12}$, $g_{23}$, $g_{31}$ et leurs inverses $g_{21}$, $g_{32}$, $g_{13}$ qui opèrent nécessairement comme $\alpha_{12}$, $\alpha_{23}$, $\alpha_{31}$ et leurs inverses $\alpha_{21}$, $\alpha_{32}$, $\alpha_{13}$ en vertu de (10.2). On va montrer que

$$\gamma_{123} = g_{12}g_{23}g_{31};$$

cela résulte de

$$\begin{aligned}\gamma_{123} &= \gamma_{124}\cdot\alpha_{14}(\gamma_{423})\cdot\gamma_{143} = \gamma_{124}\cdot\alpha_{14}\alpha_{42}(\gamma_{234})\cdot\gamma_{143}\\ &= g_{12}g_{24}g_{41}\cdot g_{14}\cdot g_{42}\cdot g_{23}\cdot g_{34}\cdot g_{42}\cdot g_{24}\cdot g_{41}\cdot g_{14}\cdot g_{43}\cdot g_{31}\\ &= g_{12}\cdot g_{23}\cdot g_{31}.\end{aligned}$$

La suite de la démonstration est une conséquence facile de l'alternation. Par exemple

$$\gamma_{412} = \alpha_{41}(\gamma_{124}) = g_{41}\cdot g_{12}g_{24}g_{41}\cdot g_{14} = g_{41}\times g_{12}\times g_{24}, \text{ etc.} \ldots.$$

C.q.f.d.

**12. Remarque sur la classe nulle.** Par définition la classe nulle de $H^2(\mathfrak{U}, \Phi)$ est formée des 2-cocycles réduits alternés $z = (\alpha_{ij}, \gamma_{ijk})$ tels qu'il existe $\xi = (\xi_i) \in C^0(\mathfrak{U}, \mathfrak{A})$ et $y = (g_{ij}) \in C_a^1(\mathfrak{U}, \mathfrak{G})$ tels que

(12.1) $$\xi * z = \delta y \qquad \text{ou} \qquad z = \xi^{-1} * \delta y.$$

Le 2-cocycle $z$ n'est donc pas nécessairement un cobord. Toutefois la propriété suivante, à mettre en rapport avec la proposition 8.1, montre qu'un 2-cocycle vérifiant (12.1) et un 2-cobord sont "pratiquement" la même chose.

PROPOSITION 12.1. *Si le 2-cocycle $z$ appartient à la classe nulle il existe* (*a*) *un recouvrement* $\mathfrak{W} = (W_r)_{r \in R}$ *tel que* $\mathfrak{U}$ *soit plus fin que* $\mathfrak{W}$; (*b*) *un 2-cocycle* $\bar{z} \in Z_a^2(\mathfrak{W}, \Phi)$ *et une* 1*-cochaîne* $y \in C_a^1(\mathfrak{U}, \mathfrak{G})$; (*c*) *des applications admissibles* $\phi: R \to I$, $\phi': R \to I$ *telles que*

$$\phi(\bar{z}) = z, \tag{1}$$

$$\phi'(\bar{z}) = \delta y. \tag{2}$$

*Démonstration.* Définissons $R$ comme la réunion de $I$ et d'une copie disjointe $I'$ dont les éléments seront notés $i', j', k' \ldots$. Ensuite on pose $W_i = W_{i'} = U_i$. Pour définir $\bar{z} = (\bar{\alpha}_{rs}, \bar{\gamma}_{rst})$ on utilise les cochaînes $\xi$ et $y$ intervenant dans (12.1) et on pose

$$
\begin{gathered}
y = (g_{ij}),\ \rho_{ij} = \rho(g_{ij}) \\
\bar{\alpha}_{ij} = \alpha_{ij}, \quad \bar{\alpha}_{i'j'} = \rho_{ij}, \quad \bar{\alpha}_{i'i} = \xi_i, \quad \bar{\alpha}_{i'j} = \alpha_{i'i}\alpha_{ij} = \xi_i\alpha_{ij}, \\
\alpha_{ii'} = \xi_i^{-1}, \quad \bar{\alpha}_{ij'} = \alpha_{ij}\alpha_{jj'} = \alpha_{ij}\xi_j^{-1}, \\
\bar{\gamma}_{ijk} = \gamma_{ijk}, \quad \bar{\gamma}_{i'j'k'} = (\delta g)_{ijk}, \\
\bar{\gamma}_{i'jk} = \bar{\gamma}_{i'j'k} = \bar{\gamma}_{i'jk'} = \xi_i(\gamma_{ijk}), \\
\bar{\gamma}_{ij'k} = \bar{\gamma}_{ijk'} = \bar{\gamma}_{ij'k'} = \gamma_{ijk}.
\end{gathered}
$$

Il faut d'abord vérifier que $\bar{z}$ est une 2-cochaîne (condition (4.2)), ce qui résulte du fait que $z$ et $\delta y$ en sont et des égalités (cfr. condition (2.2))

$$
\begin{aligned}
\rho(\bar{\gamma}_{i'jk}) &= \rho[\xi_i(\gamma_{ijk})] = \xi_i \quad \rho(\gamma_{ijk})\xi_i^{-1} = \xi_i \quad \alpha_{ij}\alpha_{jk}\alpha_{ki} \quad \xi_i^{-1} \\
&\qquad = \bar{\alpha}_{i'j}\bar{\alpha}_{jk}\bar{\alpha}_{ki'}, \\
\rho(\bar{\gamma}_{i'j'k}) &= \ldots = \bar{\alpha}_{i'j'}\bar{\alpha}_{j'k}\bar{\alpha}_{ki'}, \\
\rho(\bar{\gamma}_{ijk'}) &= \ldots = \bar{\alpha}_{ij}\bar{\alpha}_{jk'}\bar{\alpha}_{k'i}, \\
\rho(\bar{\gamma}_{ij'k}) &= \alpha_{ij}\alpha_{jk}\alpha_{ki} = \bar{\alpha}_{ij'}\bar{\alpha}_{j'k}\bar{\alpha}_{ki} \\
\rho(\bar{\gamma}_{ijk'}) &= \ldots \qquad = \bar{\alpha}_{ij}\bar{\alpha}_{jk'}\bar{\alpha}_{k'i} \\
\rho(\bar{\gamma}_{ij'k'}) &= \ldots \qquad = \bar{\alpha}_{ij'}\bar{\alpha}_{j'k'}\bar{\alpha}_{k'i}.
\end{aligned}
$$

Nous laissons au lecteur le soin de vérifier que $\bar{z}$ est alternée (définition (5.2). Il faut ensuite vérifier que $\bar{z}$ est un cocycle (condition (6.2)), ce qui résulte de vérifications faciles; par exemple:

$$
\begin{aligned}
\bar{\gamma}_{i'jk} &= \xi_i(\gamma_{ijk}); \\
\bar{\gamma}_{i'jl}\cdot\bar{\alpha}_{i'l}(\bar{\gamma}_{ljk})\cdot\bar{\gamma}_{i'lk} &= \xi_i(\gamma_{ijl})\cdot\xi_i\alpha_{il}(\gamma_{ljk})\cdot\xi_i(\gamma_{ilk}) \\
&= \xi_i(\gamma_{ijl}\cdot\alpha_{il}(\gamma_{ljk})\cdot\gamma_{ilk}) = \xi_i(\gamma_{ijk}); \\
\bar{\gamma}_{i'jl'}\cdot\bar{\alpha}_{i'l'}(\gamma_{l'jk})\cdot\gamma_{i'l'k} &= \xi_i(\gamma_{ijl})\cdot\xi_i\alpha_{il}\xi_j^{-1}\xi_j(\gamma_{ljk})\cdot\xi_i(\gamma_{ilk}) \\
&= \xi_i(\gamma_{ijl}\cdot\alpha_{il}(\gamma_{ljk})\cdot\gamma_{ilk}) = \xi_i(\gamma_{ijk}).
\end{aligned}
$$

Les propriétés (1) et (2) sont alors immédiates si l'on définit $\phi$ et $\phi'$ au moyen de

$$\phi(i) = i \qquad \phi'(i) = i'. \qquad \text{C.q.f.d.}$$

## Bibliographie


1. P. Dedecker, *Jets locaux, faisceaux, germes de sous-espaces*, Bull. Soc. Math. Belg., *6* (1953–4), 119.
2. ——— *Extension du groupe structural d'un espace fibré*, Colloque de topologie de Strasbourg (mai 1955).
3. ——— *Cohomologie à coefficients non abéliens et espaces fibrés*, Bull. Acad. Roy. Belg., *41* (1955), 1132–1146.
4. ——— *La structure algébrique de l'ensemble des classes fibrés*, Bull Acad. Roy. Belg., *42* (1956) 270–290.
5. ——— *Groupoïdes de cohomologie à coefficients non abéliens et espaces fibrés*, Colloque de topologie algébrique, Louvain, C.B.R.M. (1956).
6. ——— *On the exact cohomology sequence of a space with coefficients in a non abelien sheaf*. International Symposium on Algebraic Topology, Mexico (1956), à paraître.
7. ——— *Cohomologie de dimension 2 à coefficients non abéliens*, Comptes rendus, Paris, *247* (1958).
8. J. Frenkel, *Sur une classe d'espaces fibrés analytiques*, Comptes rendus, Paris, *236* (1953), 40–41.
9. ——— *Cohomologie à coefficients dans un faisceau non abélien*, Comptes rendus, Paris, *240* (1957), 2368–2370.
10. ——— *Cohomologie non abélienne et espaces fibrés*, Bull. Soc. Math. France, *85* (1957), 135–220.
11. A. Grothendieck, *A general theory of fibre spaces with structure sheaf*, University of Kansas, N.S.F. Report No. 4 (August, 1955).


*Université de Liège*
*Istituto Matematico Dell' Università di Roma*

# ON HOMEOMORPHISMS BETWEEN EXTENSION SPACES

BERNHARD BANASCHEWSKI

**Introduction.** In this note, conditions are obtained which will ensure that two topological spaces are homeomorphic when they have homeomorphic extension spaces of a certain kind. To discuss this topic in suitably general terms, an unspecified extension procedure, assumed to be applicable to some class of topological spaces, is considered first, and it is shown that simple conditions imposed on the extension procedure and its domain of operation easily lead to a condition of the desired kind. After the general result has been established it is shown to be applicable to a number of particular extensions, such as the Stone-Čech compactification and the Hewitt $Q$-extension of a completely regular Hausdorff space, Katětov's maximal Hausdorff-closed extension of a Hausdorff space, the maximal zero-dimensional compactification of a zero-dimensional space, the maximal Hausdorff-minimal extension of a semi-regular space, and Freudenthal's compactification of a rim-compact space. The case of the Hewitt $Q$-extension was first discussed by Heider **(6)**.

**1. Preliminaries.** A pair $(\Gamma, \gamma)$ consisting of a class $\Gamma$ of Hausdorff spaces and a mapping $\gamma : \Gamma \to \Gamma$ such that $\gamma X$ contains $X$ as a dense subspace for each $X \in \Gamma$ will be called an *extension structure*. An extension structure $(\Gamma, \gamma)$ will be called *normal* if for any $X \in \Gamma$ the subspaces $X - \{a\}$, $a \in X$, of $X$ also belong to $\Gamma$ and $\gamma$ has the properties:

$C1$. Any sequence of dense imbeddings

$$X \xrightarrow{f} Y \xrightarrow{g} \gamma X \qquad (X, Y \in \Gamma)$$

such that $g \circ f$ is the identity mapping on $X$ can be extended to a sequence of homeomorphisms

$$\gamma X \xrightarrow{f^\gamma} \gamma Y \xrightarrow{g^\gamma} \gamma X.$$

$C2$. For each $a \in X$ there exists a dense imbedding

$$h_a : \gamma X - \{a\} \to \gamma(X - \{a\})$$

which induces the identity mapping on $X - \{a\}$.

It follows easily from $C1$ that the extension space $\gamma X$ is a topological invariant of $X$ in the sense that any homeomorphism $X \to Y$ can be extended to a homeomorphism $\gamma X \to \gamma Y$. Also, one has

LEMMA 1. *$\gamma(\gamma X) = \gamma X$ for any $X \in \Gamma$.*

Received November 21, 1958.

*Proof*. Consider the sequence of dense imbeddings

$$X \xrightarrow{j} \gamma X \xrightarrow{i} \gamma X$$

where $j$ is the natural injection of $X$ into $\gamma X$ and $i$ the identity mapping. Then, $C1$ gives, as an extension of this, the sequence of homeomorphisms

$$\gamma X \xrightarrow{j^\gamma} \gamma(\gamma X) \xrightarrow{i^\gamma} \gamma X.$$

Now, if $u \in \gamma(\gamma X) - \gamma X$ and $v = (j^\gamma)^{-1}u \in \gamma X$ then $v = (i^\gamma \circ j^\gamma)v = i^\gamma u$, but also $v = iv = i^\gamma v$ and hence $i^\gamma u = i^\gamma v$ which leads to $u = v \in \gamma X$, a contradiction.

In view of Lemma 1 one can say that $C1$ implies a certain *minimality* of the extension space $\gamma X$ of $X$: If $Y \in \Gamma$ is such that $\gamma Y = Y$ and $X \subseteq Y \subseteq \gamma X$ then $Y = \gamma X$, and thus $\gamma X$ is minimal in the class of all spaces $Y \supseteq X$, $Y \in \Gamma$, with $\gamma Y = Y$.

For a number of known normal extension structures $(\Gamma, \gamma)$ the following further condition is found to hold:

$C3$. Any dense imbedding $f : X \to Y (X, Y \in \Gamma)$ can be extended to a continuous mapping $f^\gamma$ of $\gamma X$ onto $\gamma Y$.

Such extension structures will be called *strongly normal*. Strong normality can usually be checked by means of

LEMMA 2. *$C3$ and Lemma 1 together imply $C1$.*

*Proof*. If $f : X \to Y$ and $g: Y \to \gamma X$ are dense imbeddings such that $g \circ f$ is the identity mapping on $X$ then the extensions

$$f^\gamma : \gamma X \to \gamma Y \quad \text{and} \quad g^\gamma : \gamma Y \to \gamma(\gamma X) = \gamma X$$

given by $C3$ are necessarily homeomorphisms since $g^\gamma \circ f^\gamma$ is the identity mapping on $\gamma X$.

The conditions of strong normality imply a certain *maximality* of the extension space $\gamma X$ of $X$, which complements the above mentioned minimality: If $X$ is a dense subspace of $Y$ with $\gamma Y = Y (X, Y \in \Gamma)$, then the natural injection $X \to Y$ can be extended to a continuous mapping of $\gamma X$ onto $Y$ in the case of strongly normal $(\Gamma, \gamma)$.

**2. The principal result.** Let $(\Gamma, \gamma)$ be a normal extension structure throughout this section and denote by $\gamma_0 X (X \in \Gamma)$ the subspace (not necessarily belonging to $\Gamma$) of $X$ consisting of all those $a \in X$ for which the natural injection $i : X - \{a\} \to X$ cannot be extended to a homeomorphism of $\gamma(X - \{a\})$ onto $\gamma X$. These subspaces $\gamma_0 X$ have the following basic property:

LEMMA 3. *For any sequence of dense imbeddings*

$$X \xrightarrow{f} Y \xrightarrow{g} \gamma X \qquad (X, Y \in \Gamma)$$

*such that $g \circ f$ is the identity mapping on $X$ one has $f(\gamma_0 X) = \gamma_0 Y$.*

*Proof*. First, it will be shown that $\gamma_0 Y \subseteq fX$. For any $c \in Y - fX$, the given sequence of dense imbeddings leads to a new such sequence

$$X \xrightarrow{h} Y - \{c\} \xrightarrow{k} \gamma X \qquad (h, k \text{ induced by } f, g)$$

which has the extension

$$\gamma X \xrightarrow{h^\gamma} \gamma(Y - \{c\}) \xrightarrow{k^\gamma} \gamma X;$$

also, the given sequence itself can be extended to

$$\gamma X \xrightarrow{f^\gamma} \gamma Y \xrightarrow{g^\gamma} \gamma X.$$

It follows that $f^\gamma \circ k^\gamma$ is a homeomorphism of $\gamma(Y - \{c\})$ onto $\gamma Y$, and since it maps the dense subset $fX$ of $Y - \{c\}$ identically it extends the identity mapping $Y - \{c\} \to Y$. Hence, $c \notin \gamma_0 Y$ and thus $\gamma_0 Y \subseteq fX$.

Next, take $a \in X$ and $b = fa$. Here, one has the sequence of dense imbeddings

$$X - \{a\} \xrightarrow{h} Y - \{b\} \xrightarrow{k} \gamma X - \{a\} \xrightarrow{h_a} \gamma(X - \{a\})$$

where $h$ and $k$ are again restrictions of $f$ and $g$ respectively and $h_a$ is given by $C2$. From $C2$ and the assumption that $g \circ f$ is the identity mapping on $X$ one concludes that $(h_a \circ k) \circ h$ is the identity mapping on $X - \{a\}$; hence, $C1$ is applicable and one has a sequence of homeomorphisms

$$\gamma(X - \{a\}) \xrightarrow{h^\gamma} \gamma(Y - \{b\}) \xrightarrow{(h_a \circ k)^\gamma} \gamma(X - \{a\}). \tag{1}$$

Now, suppose $a \notin \gamma_0 X$ and let $i^*$ be the homeomorphism $\gamma(X - \{a\}) \to \gamma X$ extending the natural injection $i : X - \{a\} \to X$. Then, one obtains from (1) the sequence of homeomorphisms

$$\gamma(Y - \{b\}) \xrightarrow{(h_a \circ k)^\gamma} \gamma(X - \{a\}) \xrightarrow{i^*} \gamma X \xrightarrow{f^\gamma} \gamma Y, \tag{2}$$

which has the following effect on any point $fz$, $z \in X - \{a\}$:

$$fz \to z \to z \to fz.$$

Hence, the homeomorphism $\gamma(Y - \{b\}) \to \gamma Y$ given by (2) maps the dense subset $f(X - \{a\})$ of $Y - \{b\}$ identically, and therefore the same holds for the whole of $Y - \{b\}$. Thus, one has $b \notin \gamma_0 Y$ or $f(X - \gamma_0 X) \subseteq Y - \gamma_0 Y$, and from $\gamma_0 Y \subseteq fX$ it now follows that $\gamma_0 Y \subseteq f(\gamma_0 X)$.

Conversely, assume $b \notin \gamma_0 Y$ and let $j^*$ be the homeomorphism $\gamma(Y - \{b\}) \to \gamma Y$ extending the natural injection $j$: $Y - \{b\} \to Y$. Again, one obtains from (1) a sequence of homeomorphisms

$$\gamma(X - \{a\}) \xrightarrow{h^\gamma} \gamma(Y - \{b\}) \xrightarrow{j^*} \gamma Y \xrightarrow{g^\gamma} \gamma X, \tag{3}$$

giving a homeomorphism $\gamma(X - \{a\}) \to \gamma X$ which clearly extends the natural injection $X - \{a\} \to X$. This shows $a \notin \gamma_0 X$ which implies $f(\gamma_0 X) \subseteq \gamma_0 Y$; in all, $f(\gamma_0 X) = \gamma_0 Y$ is hereby established.

The particular case where $f : X \to Y$ is a homeomorphism and $g : Y \to \gamma X$ taken as $y \to f^{-1}y$, $y \in Y$, makes it obvious that any homeomorphism $f : X \to Y$ maps $\gamma_0 X$ homeomorphically onto $\gamma_0 Y$, that is, the subspaces $\gamma_0 X$ are topological invariants of the spaces $X$.

Lemma 3 now leads immediately to

PROPOSITION 1. *If $\Gamma_0 \subseteq \Gamma$ is the class of all $X \in \Gamma$ such that $\gamma_0 X = X$ then any homeomorphism $f : X' \to Y'$ between spaces $X'$, $Y' \in \Gamma$ such that $X \subseteq X' \subseteq \gamma X$, $Y \subseteq Y' \subseteq \gamma Y$ and $X, Y \in \Gamma_0$ is the extension of a homeomorphism $X \to Y$.*

*Proof.* One has $\gamma_0 X = \gamma_0 X'$, $\gamma_0 Y = \gamma_0 Y'$ and $f(\gamma_0 X') = \gamma_0 Y'$ from Lemma 3 and hence from the given equations $X = \gamma_0 X$ and $Y = \gamma_0 Y$ also $fX = Y$.

**3. Characterization of $\Gamma_0$ for strongly normal** $(\Gamma, \gamma)$. Proposition 1 naturally leads to the question whether there exist other subclasses of $\Gamma$, $(\Gamma, \gamma)$ being any normal extension structure, for which the analogous proposition holds. Of course, this is trivially so for any subclass of $\Gamma_0$. Similarly, there may exist trivial enlargements of $\Gamma_0$ with this property: If, for instance, $\Gamma$ contains an $X$ such that $\gamma X = X$ but $\gamma_0 X' \neq X'$ for any dense subspace of $X$, then $\Gamma_0$ does not contain any space homeomorphic to $X$, and $\Gamma_0 \cup \{X\}$ would be of the said type. Consequently, one has to look for further properties of $\Gamma_0$ which together with Proposition 1 will give rise to some characterization of $\Gamma_0$. This will be done in the present section at least for the case of strongly normal extension structures $(\Gamma, \gamma)$.

A preliminary result is:

LEMMA 4. *If $(\Gamma, \gamma)$ is a strongly normal extension structure and $f : X \to Y$ $(X, Y \in \Gamma)$ a dense imbedding then $(f^\gamma)^{-1} fX = X$.*

*Proof.* Let $a \in X$ and $b \in \gamma X - X$ be such that $fa = f^\gamma b$. This has to be shown to lead to a contradiction. Since $\gamma X$ is Hausdorff there exist disjoint neighbourhoods $U$ and $V$ of $a$ and $b$ respectively in $\gamma X$. Also, since $f$ is an imbedding there exists a neighbourhood $W$ of $fa$ in $\gamma Y$ such that $f(U \cap X) = W \cap fX$. Now, by the continuity of $f^\gamma$ there exists a neighbourhood $V_0 \subseteq V$ of $b$ in $\gamma X$ such that $f^\gamma V_0 \subseteq W$. This implies $f(V_0 \cap X) \subseteq W \cap fX = f(U \cap X)$ and therefore $V_0 \cap X \subseteq U \cap X$ since $f$ is one-to-one; this, however, leads to the contradiction $\phi \neq V_0 \cap X \subseteq U \cap V = \phi$.

The desired property of $\Gamma_0$ will be obtained from the following

LEMMA 5. *If $(\Gamma, \gamma)$ is a strongly normal extension structure then $\gamma_0 X - \{a\} \subseteq \gamma_0(X - \{a\})$ for any $a \in X$, $X \in \Gamma$.*

*Proof.* Let $b \in X - \{a\}$ be such that there exists a homeomorphism $h$:

$\gamma(X - \{a, b\}) \to \gamma(X - \{a\})$ inducing the identity mapping on $X - \{a, b\}$. Then $b$ is not isolated in $X$ and one has the following diagram of continuous mappings

$$\begin{array}{ccc} \gamma(X - \{b\}) - \{a\} & \xrightarrow{g} & \gamma(X - \{a, b\}) \\ k \downarrow & & \downarrow h \\ \gamma X - \{a\} & \xrightarrow[j]{} & \gamma(X - \{a\}) \end{array}$$

where $g$ and $j$ are the imbeddings given by $C2$ and $k$ is the restriction of the continuous extension $i^\gamma$ of the natural injection $i$: $X - \{b\} \to X$ by $C3$ which is known to map $\gamma(X - \{b\}) - \{a\}$ onto $\gamma X - \{a\}$ by Lemma 4. Since all mappings induce the identity on $X - \{a, b\}$ the diagram is commutative. It follows that $k$ must be one-to-one since $g$ and $h$ are, and consequently $i^\gamma$: $\gamma(X - \{b\}) \to \gamma X$ is also one-to-one. This means that $i^\gamma$ has an inverse $f$, and this coincides with the $h_b$ (of $C2$) on $\gamma X - \{b\}$ as one sees immediately from the sequence

$$\gamma X - \{b\} \xrightarrow{h_b} \gamma(X - \{b\}) \xrightarrow{i^\gamma} \gamma X.$$

This shows that the restriction of $f$ to $\gamma X - \{b\}$ is continuous. On the other hand, the restriction of $f$ to $\gamma X - \{a\}$ is merely $k^{-1}$. Now, $h^{-1} \circ j$ and $g$ map $\gamma X - \{a\}$ and $\gamma(X - \{b\}) - \{a\}$ respectively onto the same subspace of $\gamma(X - \{a, b\})$; thus $g^{-1}$ is defined at each $h^{-1}(jx)$, $x \in \gamma X - \{a\}$, and clearly $k^{-1}x = g^{-1}(h^{-1}(jx))$. Since $h^{-1} \circ j$ and $g$ are dense imbeddings it follows that $k^{-1}$ is continuous. Therefore, the restriction of $f$ to $\gamma X - \{a\}$ is also continuous, and this shows $f$ to be continuous, hence $i^\gamma$ to be a homeomorphism and finally $b \notin \gamma_0 X$.

With this it is proved that $b \notin \gamma_0(X - \{a\})$ implies $b \notin \gamma_0 X$ which immediately gives the desired result $\gamma_0 X - \{a\} \subseteq \gamma_0(X - \{a\})$.

After these preparations, the following characterization of $\Gamma_0$ can easily be established.

Proposition 2. *If* $(\Gamma, \gamma)$ *is a strongly normal extension structure then* $X \in \Gamma_0$ *implies* $X - \{a\} \in \Gamma_0$ *for any* $a \in X$ *and* $\Gamma_0$ *is the largest class of spaces* $X \in \Gamma$ *for which this condition and Proposition* 1 *hold.*

*Proof.* By Lemma 5, $\gamma_0 X = X$ implies $X - \{a\} = \gamma_0 X - \{a\} \subseteq \gamma_0(X - \{a\})$ and thus $\gamma_0(X - \{a\}) = X - \{a\}$, as stated. Now, let $\Gamma_1$ be any subclass of $\Gamma$ such that $X - \{a\} \in \Gamma_1$ for any $X \in \Gamma_1$, $a \in X$, and Proposition 1 holds for $\Gamma_1$ (in place of $\Gamma_0$). Then, if $X \in \Gamma_1$ does not belong to $\Gamma_0$ there must be an $a \in X$ such that the natural injection $X - \{a\} \to X$ can be extended to a homeomorphism $\gamma(X - \{a\}) \to \gamma X$. However, this homeomorphism clearly does not induce a homeomorphism $X - \{a\} \to X$, and this contradicts the assumptions for $\Gamma_1$. It follows that $\Gamma_1 \subseteq \Gamma_0$.

*Remark.* We do not know whether Proposition 2 might not be true for *any* normal extension structure $(\Gamma, \gamma)$. It is clear that $\Gamma_0$ is characterized as above

whenever $X - \{a\} \in \Gamma_0$ for all $a \in X$, $X \in \Gamma_0$, and even for those $(\Gamma, \gamma)$ considered below which are not (or not known to be) strongly normal this can actually be verified explicitly.

With this, the general considerations are concluded and the following sections deal with their application to particular extension structures, that is, with the proofs of the normality or strong normality and the explicit descriptions of $\Gamma_0$ for several instances of $(\Gamma, \gamma)$. For the latter, it is useful to observe that for any normal $(\Gamma, \gamma)$ and $X \in \Gamma$ the points $a \notin \gamma_0 X$ of $X$ cannot be isolated simply because the inverse of the supposed homeomorphism $\gamma(X - \{a\}) \to \gamma X$ maps $a \in X$ into a point of $\gamma(X - \{a\}) - (X - \{a\})$ and any neighbourhood of such a point must meet $X - \{a\}$.

**4. Stone-Čech compactifications and Hewitt Q-extensions.** Let $(\mathrm{B}, \beta)$ be the extension structure for which B is the class of all completely regular Hausdorff spaces and $\beta X$ the Stone-Čech compactification of $X \in \mathrm{B}$. Obviously, $X \in \mathrm{B}$ implies $X - \{a\} \in \mathrm{B}$ for any $a \in X$, $\beta(\beta X) = \beta X$ holds for all $X \in \mathrm{B}$ and any dense imbedding $X \to Y$ has a continuous extension mapping $\beta X$ onto $\beta Y$ by the well-known maximality property of $\beta X$.

To obtain, for each $a \in X$, an imbedding of $\beta X - \{a\}$ into $\beta(X - \{a\})$ as described in *C*3 it is sufficient to prove that every bounded continuous real function $f$ on $X - \{a\}$ can be continuously extended to $\beta X - \{a\}$ **(7)**. To show this, let $u \in \beta X - X$ be any point, $V$ and $W$ disjoint closed neighbourhoods in $\beta X$ of $u$ and $a$ respectively and $g$ a continuous function on $\beta X$ such that $gV = \{1\}$ and $gW = \{0\}$. Now, with the restriction $h$ of $g$ to $X - \{a\}$ the product $fh$ is continuous on $X - \{a\}$ and vanishes on $W \cap (X - \{a\})$; it can therefore be extended continuously to $X$ with value 0 at $a$, and the resulting function has a continuous extension $f^*$ to $\beta X$. $f^*$ satisfies $f^*x = fx \cdot hx = fx$ for all $x \in V \cap X$, hence

$$\lim_{\substack{x \to u \\ x \in X - \{a\}}} fx = \lim_{\substack{x \to u \\ x \in X}} f^*x = f^*u.$$

The existence of this limit for any $u \in \beta X - \{a\}$ implies **(4**, ch. I, § 6**)** that $f$ can be continuously extended to $\beta X - \{a\}$.

In all, it is then established that $\beta$ satisfies the conditions *C*1—*C*3. Further, it is obvious that for any non-isolated $a \in X$ the extension of the natural injection $X - \{a\} \to X$ to $\beta(X - \{a\})$ is a homeomorphism if and only if any bounded continuous real function on $X - \{a\}$ has a continuous extension to $X$. Hence one has:

PROPOSITION 3. *The extension structure* $(\mathrm{B}, \beta)$ *is strongly normal and* $\mathrm{B}_0$ *is the class of all* $X \in \mathrm{B}$ *such that for any non-isolated* $a \in X$ *there exist bounded continuous real functions on* $X - \{a\}$ *without a continuous extension to* $X$.

*Remark.* The class $\mathrm{B}_0$ can also be described in a variety of other ways such as: the filter on $X - \{a\}$ given by the sets $V - \{a\}$, $V$ the neighbourhoods

of $a$ in $X$, is not maximal completely regular (**4**, ch. IX, p. 15) on $X - \{a\}$ for any $a \in X$; or, as in (**6**), the finite open normal coverings of $X - \{a\}$ are not all induced by such coverings of $X$. In the case of particular types of spaces $X$, simpler topological conditions can be given. Thus, for fully normal $X$, $X \in B_0$ is equivalent to the condition that $\{a\} = A \cap B$ with closed sets $A$, $B$ both different from $\{a\}$, for each $a \in X$. Similarly, for locally compact $X$, $X \in B_0$ amounts to the existence of compact $A, B \subseteq X$ with $A \cap B = \{a\}$, $A$ and $B$ different from $\{a\}$, for each $a \in X$.

If one considers the Hewitt $Q$-extension $\upsilon X$ for completely regular Hausdorff spaces $X$, one can easily see, by essentially the same argument as above, that any continuous real function on $X - \{a\}$ can be extended continuously to $\upsilon X - \{a\}$. Using the standard properties of $\upsilon X$, one then obtains for the class $\Upsilon$ of completely regular Hausdorff spaces and the operation $\upsilon$ of Hewitt $Q$-extension:

PROPOSITION 4. *The extension structure* $(\Upsilon, \upsilon)$ *is normal and* $\Upsilon_0$ *is the class of all* $X \in Y$ *such that for any* $a \in X$ *there exist continuous real functions on* $X - \{a\}$ *without a continuous extension to* $X$.

*Remark* 1. Proposition 1 for $(\Gamma, \gamma) = (\Upsilon, \upsilon)$ which is hereby established is due to Heider (**6**) in the case where $X' = \upsilon X$ and $Y' = \upsilon Y$.

*Remark* 2. Although $(\Upsilon, \upsilon)$ is normal it is not strongly normal since a space $E$ may have extension spaces $X$ and $Y$ such that $E \subseteq X \subset Y \subseteq \beta E$, $\upsilon X = X$ and $\upsilon Y = Y$, in which case $C3$ breaks down for the natural injection $X \to Y$. Nevertheless, it is still true that $X \in \Upsilon_0$ implies $X - \{a\} \in \Upsilon_0$ for any $a \in X$ and hence the characterization of $\Upsilon_0$ as in Proposition 2 still applies.

**5. Katětov extensions.** For any Hausdorff space $X$ which is not absolutely closed, Katětov (**7**) introduced an absolutely closed extension which can be described as follows: Corresponding to each non-convergent maximal open (that is, with a basis consisting of open sets) filter $\mathfrak{M}$ on $X$ a new point $x_{\mathfrak{M}}$ is adjoined to $X$ and on this enlarged set the collection of all sets $V \cup \{x_{\mathfrak{M}}\}$, $V$ open in $X$ and $V \in \mathfrak{M}$, is taken as a basis for the open sets. If one considers on the class K of all Hausdorff spaces the operator $\kappa$ which assigns to each $X \in$ K its Katětov extension $\kappa X$ ($\kappa X = X$ if $X$ absolutely closed), one has:

PROPOSITION 5. (K, $\kappa$) *is a strongly normal extension structure and* $K_0$ *is the class of all* $X \in$ K *such that for any non-isolated* $a \in X$ *there exists an open* $U \subseteq X$ *such that* $a \in \bar{U}$ *but* $U \cup \{a\}$ *is not open.*

*Proof.* $\kappa(\kappa X) = \kappa X$ and the basic properties of $\kappa X$ proved in (**7**) implies $C1$ and $C3$ for (K, $\kappa$). $C2$ follows from the fact that there is a natural one-to-one correspondence between the maximal non-convergent open filters on $X$ and those filters of this kind on $X - \{a\}$ which do not converge to $a$ in $X$ given by $\mathfrak{M} \to \{A | a \notin A \in \mathfrak{M}\}$.

To obtain the stated description of $K_0$, one first observes that $a \notin \kappa_0 X$ holds if and only if the filter consisting of the sets $V - \{a\}$, $V$ the neigh-

bourhoods of $a$ in $X$, is a maximal open filter in $X - \{a\}$. Next, this is obviously the case if and only if any open $U$ in $X - \{a\}$ which meets all these $V - \{a\}$ is itself one of them. Finally, in terms of the topology of $X$ itself, this condition means that for any open $U \subseteq X$ such that $a \in \bar{U} - U$, $U \cup \{a\}$ is open in $X$. Therefore, $a \in \kappa_0 X$ is equivalent, for non-isolated $a \in X$, to the existence of an open $U \subseteq X$ with $a \in \bar{U}$ for which $U \cup \{a\}$ is not open.

*Remark.* Proposition 1 for $(\Gamma, \gamma) = (\mathrm{K}, \kappa)$ which is thus obtained was proved by Katětov **(7)**.

**6. Maximal zero-dimensional compactifications.** For any zero-dimensional Hausdorff space $X$ there is defined a maximal zero-dimensional compact extension $\zeta X$ which can be considered as the completion of $X$ with respect to the uniform structure of $X$ given by the finite partitions of $X$ into open-closed sets **(1)**. Alternatively, $\zeta X$ is the maximal ideal space of the Boolean algebra of all open-closed sets in $X$, $X$ imbedded in this as usual by identifying each point with the corresponding fixed ideal. Yet another description of $\zeta X$ is as follows: Let $\Phi_\zeta(X)$ be the set of all maximal open-closed filters $\mathfrak{M}$ (that is, with a basis consisting of open-closed sets) in $X$ with void adherence. Then, corresponding to each $\mathfrak{M} \in \Phi_\zeta(X)$ a new point $x_{\mathfrak{M}}$ is adjoined to X, and on this enlarged set the collection of all sets $V \cup \{x_{\mathfrak{M}} | V \in \mathfrak{M} \in \Phi_\zeta(X)\}$, $V$ open-closed in $X$, is taken as a basis for the open sets.

If $Z$ denotes the class of all zero-dimensional Hausdorff spaces and $\zeta$ the operator which assigns to each $X \in Z$ its extension $\zeta X$ one has:

PROPOSITION 6. *$(\mathrm{Z}, \zeta)$ is a strongly normal extension structure and $\mathrm{Z}_0$ is the class of all $X \in \mathrm{Z}$ such that for any non-isolated $a \in X$ there exists an open $U \subseteq X$ for which $\bar{U} = U \cup \{a\}$ and $U \cup \{a\}$ is not open.*

*Proof.* One has $\zeta(\zeta X) = \zeta X$ for any $X \in \mathrm{Z}$ and $C3$ follows immediately from the maximality property of $\zeta X$ which states **(1)** that any zero-dimensional compact extension of $X$ is the continuous image of $\zeta X$ under a mapping which extends the identity mapping on $X$. $C2$ is again obtained from the fact that (i) $\mathfrak{M} \to \mathfrak{M}_* = \{A | a \notin A \in \mathfrak{M}\}$ is a one-to-one correspondence between the $\mathfrak{M} \in \Phi_\zeta(X)$ and those $\mathfrak{N} \in \Phi_\zeta(X - \{a\})$ which do not converge to $a$ and (ii) $V \in \mathfrak{M}$ is equivalent to $V - \{a\} \in \mathfrak{M}_*$.

As to the characterization of $\mathrm{Z}_0$, it follows immediately from the definition of $\zeta(X - \{a\})$ in terms of maximal open-closed filters that $a \notin \zeta_0 X$ holds if and only if the open-closed filter on $X - \{a\}$ consisting of the sets $V - \{a\}$, $V$ the neighbourhoods of $a$ in $X$, is a maximal open-closed filter in $X - \{a\}$. This will be the case if and only if any open-closed $W$ in $X - \{a\}$ which meets all these $V - \{a\}$ is itself one of them. Expressed in terms of the topology of $X$ this means that for any open $W \subseteq X$ with $\bar{W} = W \cup \{a\}$ the set $W \cup \{a\}$ is open. Therefore, $a \in \zeta_0 X$ holds for non-isolated $a \in X$ exactly if there exists an open $U \subseteq X$ with $\bar{U} = U \cup \{a\}$ for which $U \cup \{a\}$ is not open.

A consequence of Proposition 1 for $(\Gamma, \gamma) = (Z, \zeta)$, obtained from the description of $\zeta X$ in terms of the Boolean algebra $B(X)$ of the open-closed sets of $X$, is the following:

COROLLARY. *If $X$, $Y \in Z_0$, then any isomorphism $B(X) \to B(Y)$ is induced by a homeomorphism $X \to Y$.*

**7. Maximal Hausdorff-minimal extensions.** A space $X$ is called Hausdorff-minimal if it is Hausdorff and its topology is minimal in the partially ordered set of all Hausdorff topologies on $X$ (order by inclusion between the collections of open sets); in other words, if any continuous one-to-one mapping of $X$ is a homeomorphism. It is known **(3)** that any semi-regular space $X$ (that is, $X$ is Hausdorff and the interiors of closed sets in $X$ form a basis for the open sets) possesses a Hausdorff-minimal extension $\sigma X$ such that for any Hausdorff-minimal extension $W \supseteq X$ ($X$ dense in $W$), the natural injection $X \to W$ can be extended to a continuous mapping of $\sigma X$ onto $W$. A description of $\sigma X$ can be given as follows: Let a filter in $X$ be called semi-regular if it has a basis consisting of regular open sets (= the interiors of closed sets) and denote by $\Phi_\sigma(X)$ the set of all maximal semi-regular filters in $X$ whose adherence is void. Then, adjoin to $X$ a new point $x_{\mathfrak{M}}$ for each $\mathfrak{M} \in \Phi_\sigma(X)$ and take as basis for the open sets on this enlarged set the collection of all sets $V \cup \{x_{\mathfrak{M}} | V \in \mathfrak{M} \in \Phi_\sigma(X)\}$ where $V$ is a regular open set in $X$.

Now, if $\Sigma$ denotes the class of all semi-regular spaces and $\sigma$ the operator which assigns to each $X \in \Sigma$ its extension $\sigma X$ one has:

PROPOSITION 7. *$(\Sigma, \sigma)$ is a strongly normal extension structure and $\Sigma_0$ is the class of all $X \in \Sigma$ such that any non-isolated point of $X$ belongs to the closures of two disjoint open sets.*

*Proof.* The strong normality follows from the mentioned properties of $\sigma X$ in the same way as it was obtained for $(Z, \zeta)$. To determine $\Sigma_0$ one again observes that $a \in \sigma_0 X$ ($a \in X$ non-isolated) means that the semi-regular filter on $X - \{a\}$ consisting of all $V - \{a\}$, $V$ the neighbourhoods of $a$ in $X$, is not a maximal such filter on $X - \{a\}$. Since finite intersections of regular open sets are regular open, this means that there exists a regular open set $W$ of $X - \{a\}$ which meets all $V - \{a\}$ but is not itself one of them, that is, for which $W \cup \{a\}$ is not open. Such a $W$ is the interior, in $X - \{a\}$, of its closure $\bar{W} - \{a\}$ in $X - \{a\}$, and since $X - \{a\}$ is open in $X$ this is the same as saying that $W$ is the interior of $\bar{W} - \{a\}$ in $X$. Also, since $W \cup \{a\}$ is not open $W$ is merely the interior of $\bar{W}$ in $X$, that is, a regular open set of $X$, and $a$ belongs to $\bar{W}$ as well as to the closure of its complement. Therefore, $a \in \bar{U} \cap \bar{W}$ with open disjoint $U$ and $W$. Conversely, if this is the case then $a$ cannot belong to the interior $W_0$ of $\bar{W}$ since $W_0 \cap U = \phi$, and hence $W_0 \subseteq X - \{a\}$ is the interior, in $X - \{a\}$, of the closed set $\bar{W} - \{a\}$ in $X - \{a\}$, that is, a regular open set of $X - \{a\}$. Also, $a \in \bar{W} = \bar{W}_0$ shows

that $W_0$ meets all $V - \{a\}$. Finally, $W_0 \cup \{a\}$ is not open, for if it were $a \in \bar{W}$ would imply $a \in W_0$.

**8. Freudenthal extensions.** A Hausdorff space $X$ is called rim-compact (also: semi-compact) if the open sets $V \subseteq X$ whose boundary $B(V) = \bar{V} \cap CV$ ($CV$ the complement of $V$) is compact form a basis for the open sets in $X$. For any such space $X$ there exists, according to **(5)**, a compact extension $\varphi X$ with the property that (i) every point in $\varphi X$ has arbitrarily small neighbourhoods whose boundaries lie in $X$ and (ii) for any other compact extension $W \supseteq X$ satisfying (i) the natural injection $X \to W$ has a continuous extension $\varphi X \to W$. The extension $\varphi X$ has been described in the following way: If $\Phi_\varphi(X)$ denotes the set of all filters in $X$ which have a basis consisting of open sets with compact boundary (such filters will be called rim-compact here), are maximal with respect to this property and have void adherence, then $\varphi X$ is obtained by adjoining to $X$ a new point $x_{\mathfrak{M}}$ for each $\mathfrak{M} \in \Phi_\varphi(X)$ and taking the sets $V \cup \{x_{\mathfrak{M}} | V \in \mathfrak{M} \in \Phi_\varphi(X)\}$, $V \subseteq X$ open with compact $B(V)$, as basis for the open sets in this enlarged set **(5)**. Alternatively, $\varphi X$ is the completion of $X$ with respect to the uniform structure of $X$ which is defined by the finite coverings of $X$ by open sets with compact boundary **(8)**.

For the class $\Phi$ of all rim-compact Hausdorff spaces and the operator which assigns to each $X \in \Phi$ the extension $\varphi X$ the following holds:

PROPOSITION 8. $(\Phi, \varphi)$ *is a normal extension structure and* $\Phi_0$ *is the class of all* $X \in \Phi$ *such that any non-isolated* $a \in X$ *lies on the boundary* $B(U)$ *of some open* $U \subseteq X$ *for which* $B(U) - \{a\}$ *is compact and* $U \cup \{a\}$ *not open.*

*Proof.* To verify C1 for $(\Phi, \varphi)$ it is sufficient to consider $X, Y \in \Phi$ such that $X \subseteq Y \subseteq \varphi X$ and prove that the sequence of natural injections

$$X \xrightarrow{i} Y \xrightarrow{j} \varphi X$$

can be extended to a sequence of homeomorphisms

$$\varphi X \xrightarrow{i^\varphi} \varphi Y \xrightarrow{j^\varphi} \varphi X.$$

This amounts to the same as the existence of a homeomorphism $\varphi Y \to \varphi X$ which extends the natural injection $j$. Now, the existence of a continuous extension $k$ of $j$ mapping $\varphi Y$ onto $\varphi X$ follows from the maximality property of $\varphi Y$ and since $\varphi Y$ and $\varphi X$ are compact it is enough to show $k$ to be one-to-one. Therefore, consider $\mathfrak{L}, \mathfrak{M} \in \Phi_\varphi(Y)$ such that $kx_{\mathfrak{L}} = kx_{\mathfrak{M}} = x_{\mathfrak{N}}$, $\mathfrak{N} \in \Phi_\varphi(X)$. If $\mathfrak{L}_X$ and $\mathfrak{M}_X$ are the filters on $X$ obtained from $\mathfrak{L}$ and $\mathfrak{M}$ respectively by intersecting all their sets with $X$ one has $\mathfrak{N} \subseteq \mathfrak{L}_X, \mathfrak{M}_X$ by the continuity of $k$ and the limit relations $\lim \mathfrak{L} = x_{\mathfrak{L}}$, $\lim \mathfrak{M} = x_{\mathfrak{M}}$ in $\varphi Y$. Now, $\mathfrak{L}_X$ and $\mathfrak{M}_X$ are rim-compact filters: For any open $U \in \mathfrak{L}$ there exists a $V \in \mathfrak{L}$ such that $\bar{V} \subseteq U$ and $B(V)$ is compact (topological operations all in $Y$). This

$B(V)$ can be covered by finitely many open sets $W_i$ in $\varphi X$ such that $V_i = W_i \cap Y \subseteq U$ and the boundaries $B(V_i)$ are compact and lie in $X$. For the open set $V^* = V \cup \bigcup V_i$ one has $B(V^*) \subseteq B(V) \cap \bigcup B(V_i)$ which implies $B(V^*) \subseteq \bigcup B(V_i)$ since $B(V) \subseteq \bigcup V_i$ shows that no $x \in B(V)$ belongs to $B(V^*)$. It follows that $B(V^*)$ is compact and lies in $X$; hence $X \cap V^*$ is an open set with compact boundary of $X$. As $V^* \in \mathfrak{L}$ (by $V^* \supseteq V$) and $V^* \subseteq U$ where $U$ was an arbitrary open set in $\mathfrak{L}$, this means that $\mathfrak{L}_X$ is a rim-compact filter. This result now leads to the equations $\mathfrak{L}_X = \mathfrak{N} = \mathfrak{M}_X$ which in turn give $\mathfrak{L} = \mathfrak{M}$; finally, this proves that $k$ is one-to-one.

The proof of $C2$ for $(\Phi, \varphi)$ is of the same nature as that for $(\mathrm{Z}, \mathfrak{z})$, and the remaining thing is to characterize $\Phi_0$. For this one uses the fact that a rim-compact filter $\mathfrak{S}$ on a space is maximal if and only if any open $U$ with compact boundary which meets all sets of $\mathfrak{S}$ itself belongs to $\mathfrak{S}$; this can easily be deduced from the relation $B(U \cap V) \subseteq B(U) \cap B(V)$. It follows that a non-isolated $a \in X$ belongs to $\varphi_0 X$ if and only if there exists an open $U \subseteq X - \{a\}$ which meets all neighbourhoods of $a$, that is , $a \in \bar{U}$, and whose boundary in $X - \{a\}$ is compact, but for which $U \cup \{a\}$ is not open in $X$. Since $a \notin U$ and $a \in \bar{U}$ means $a \in B(U)$, the boundary of $U$ in $X$ ($U$ is open in $X$), the boundary of $U$ in $X - \{a\}$ is $B(U) - \{a\}$. This completes the proof of Proposition 8.

*Remark.* $(\Phi, \varphi)$ is not strongly normal: If $X$ is the open circular unit disc and $Y$ the closed circular unit disc in the plane and $f : X \to Y$ the natural injection there exists no extension of $f$ to a continuous mapping of $\varphi X$ onto $\varphi Y = Y$ since $\varphi X$ is the one-point compactification of $X$.

## References


1. B. Banaschewski, *Ueber nulldimensionale Räume*, Mathematische Nachrichten, *13* (1955) 129–140.
2. ———, *Ueber zwei Extremaleigenschaften topologischer Räume*, Mathematische Nachrichten, *13* (1955), 141–150.
3. ———, Report of the Summer Research Institute of the Canadian Mathematical Congress, 1957.
4. N. Bourbaki, *Topologie générale*, Actualities scientifiques et industrielles (Paris, Hermann et Cie.).
5. H. Freudenthal, *Neuaufbau der Endentheorie*, Ann. Math., *43* (1942), 261–279.
6. L. J. Heider, *A note concerning completely regular $G_\delta$-spaces*, Proc. Amer. Math. Soc., *8* (1957), 1060–1066.
7. M. Katětov, *Ueber H-abgeschlossene und bikompakte Räume*, Casopis mat. a fys., *69* (1939–40), 36–49.
8. K. Morita, *On bicompactifications of semibicompact spaces*, Sci. Rep. Tokyo Dunrika Dagaiku, Sect. A., *4* (1952), 222–229.
9. H. Wallman, *Lattices and topological spaces*. Ann. Math., *39* (1938), 112–126.


*Summer Research Institute of*
*the Canadian Mathematical Congress*

# THE COMMUTATIVITY OF A SPECIAL CLASS OF RINGS

WALLACE S. MARTINDALE, III

A well-known theorem of Jacobson **(1)** states that if every element $x$ of a ring $R$ satisfies $x^{n(x)} = x$ where $n(x) > 1$ is an integer, then $R$ is commutative. A series of generalizations of this theorem have been proved by Herstein **(2; 3; 4; 5; 6)**, his last result in this direction **(6)** being that a ring $R$ is commutative provided every commutator $u$ of $R$ satisfies $u^{(n)u} = u$. We now define a $\gamma$-ring to be a ring $R$ in which $u^{n(u)} - u$ is central for every commutator $u$ of $R$ (where $n(u) > 1$ is an integer). In the present paper we verify the following conjecture of Herstein: every commutator of a $\gamma$-ring is central.

## 1. Semi-simple $\gamma$-rings.

The main step in our paper consists in proving

THEOREM 1. *Every division $\gamma$-ring $D$ is a field.*

*Proof.* We will show that every commutator $u \in D$ satisfies $u^{m(u)} = u$ where $m(u) > 1$ is an integer. It will then follow immediately from Herstein's theorem in **(6)** that $D$ is a field.

Suppose there exists a commutator $u = xy - yx$ which does not lie in the centre $Z$. Let $C$ be the prime field of $D$, where either $C = R$ if the characteristic is zero or $C = P$ in the case of characteristic $p > 0$. We denote by $K = C(u)$ the subfield of $D$ generated by $C$ and $u$ and let $k = K \cap Z$. We remark that $k$ is a proper subfield of $K$ containing $C$, since $u \notin k$. For all $\lambda \in k$ $\lambda u = \lambda(xy-yx) = (\lambda x)y - y(\lambda x)$ is a commutator of $D$ lying in $K$. We make the important observation that $(\lambda u)^{n(\lambda)} - (\lambda u) \in k$ for all $\lambda \in k$, where $n(\lambda) > 1$ is an integer. Indeed, $(\lambda u)^{n(\lambda)} - (\lambda u) \in Z$ because $D$ is a $\gamma$-ring, and $(\lambda u)^{n(\lambda)} - (\lambda u) \in K$ since both $\lambda$ and $u \in K$.

Only three possibilities may now arise, namely,

(1) $u$ is transcendental over $C$
(2) $u$ is algebraic over $R$
(3) $u$ is algebraic over $P$.

In (1) we know by Luroth's Theorem that $k$ is a simple transcendental extension $C(t)$ of $C$. Our immediate objective is to rule out possibilities (1) and (2). In order to do so we shall require the assistance of two lemmas.

Received February 8, 1959. The material in this paper is a portion of a dissertation submitted to the University of Pennsylvania under the guidance of Dr. I. N. Herstein. The work on the paper was done at Yale University while the author was a research assistant to Dr Herstein under a National Science Foundation grant (NSF-2270).

LEMMA 1. *If* $\{\lambda_i\}$ *is an infinite sequence of distinct non-zero elements of* $k$, *then* $n(\lambda_i) \to \infty$, *where* $(\lambda_i u)^{n(\lambda_i)} - (\lambda_i u) \in k$, $i = 0, 1, 2 \ldots$.

*Proof.* If the statement were not true, there would exist an infinite subsequence $\{\lambda_j'\}$ with the property that $n(\lambda_j') = n$, a constant. Setting $\lambda_j'' = \lambda_j'(\lambda_0')^{-1}$, we can write our basic equations in the form

(a) $(\lambda_j''\lambda_0'u)^n - (\lambda_j''\lambda_0'u) = (\lambda_j'u)^n - (\lambda_j'u) \in k$, $j = 0, 1, 2 \ldots$. Multiplication of the equation $(\lambda_0'u)^n - (\lambda_0'u) \in k$ by $(\lambda_j'')^n$ gives us

(b) $(\lambda_j'')^n(\lambda_0'u)^n - (\lambda_j'')^n(\lambda_0'u) \in k$, $j = 0, 1, 2 \ldots$. Subtraction of (a) from (b) yields

(c) $[(\lambda_j'')^n - \lambda_j''](\lambda_0'u) \in k$, $j = 0, 1, 2 \ldots$.

$\lambda_0'u \notin k$ because $u \notin k$ and $\lambda_0' \neq 0$; it then follows from (c) that $(\lambda_j'')^n - (\lambda_j'') = 0$, $j = 0, 1, 2 \ldots$. A contradiction results since we now have an infinite number of distinct elements of the field $k$ satisfying the same equation $\mu^n - \mu = 0$.

LEMMA 2. *In* (1) *and* (2) *suppose that* $V$ *is non-trivial discrete non-Archimedean valuation of* $k$ *and* $W$ *an extension of* $V$ *to* $K$. *Then* $\bar{k} = \bar{K}$, *where* $\bar{k}$ *and* $\bar{K}$ *are the completions of* $k$ *and* $K$ *relative to* $V$ *and* $W$, *respectively.*

*Proof.* We begin by choosing a Cauchy sequence $\{\lambda_i\}$ of non-zero distinct elements of $k$ converging to a non-zero element $\bar{\lambda} \in \bar{k}$, where for all $i$ $V(\lambda_i) > 1 + |W(u)|\, i$. For each $\lambda_i$ of the sequence we pick an $n(\lambda_i) > 1$ such that

$$(\lambda_i u)^{n(\lambda_i)} - (\lambda_i u) = \gamma_i \in k, \quad i = 0, 1, 2 \ldots.$$

$n(\lambda_i) \to \infty$ by Lemma 1. The relationship

$$W[(\lambda_i u)^{n(\lambda_i)}] = n(\lambda_i)[V(\lambda_i) + W(u)] > n(\lambda_i)$$

then shows that the element $(\lambda_i u)^{n(\lambda_i)}$ converges to 0 in $\bar{K}$. $\lambda_i u$ converges to $\bar{\lambda}u$, $\bar{\lambda} \neq 0$. It follows that $\{\gamma_i\}$ is a Cauchy sequence in $k$ and thus must converge to an element $\bar{\gamma} \in \bar{k}$. We have now analysed all the terms of the equations

$$(\lambda_i u)^{n(\lambda_i)} - \lambda_i u = \gamma_i, \qquad i = 0, 1, 2 \ldots$$

and, letting $i \to \infty$, we can conclude that $\bar{\lambda}u = \bar{\gamma}$, $\bar{\lambda} \neq 0$. Hence $u \in \bar{k}$ and $\bar{k} = \bar{K}$.

We are now in a position to rule out the possibilities (1) and (2). $K$ is a finite separable extension of $k$ since its generator $u$ satisfies the separable polynomial $\mu^{n(1)} - \mu \in k$. In (1) we take as our set of valuations of $k$ all those which act trivially on the prime field $C$, and in (2) we consider all those which reduce to *p-adic* valuations of $R$. We shall denote the ring of integers of $k$ by $o$ and the discriminant ideal of $o$ by $d$. We let $G(V)$ stand for the value group of a valuation $V$ of $k$. If $B$ is a subgroup of a group $A$ then the index of $B$ in $A$ will be symbolized by $(A : B)$.

Lemma 2 tells us that no valuation $V$ of $k$ can ramify in $K$. Indeed, $\bar{K} = \bar{k}$

implies that $G(\bar{W}) = G(\bar{V})$, where $\bar{V}$ and $\bar{W}$ are the valuations of the completions $\bar{k}$ and $\bar{K}$ relative to $V$ and any extension $W$. It follows then that the ramification number $e = (G(W) : G(V)) = (G(\bar{W}) : G(\bar{V})) = 1$. To say that no valuation $V$ ramifies means that no prime ideal of $o$ divides the discriminant ideal $d$. Since any proper non-zero ideal of $o$ is a product of prime ideals we must assume that $d = o$. But this forces $K = k$, a contradiction. We must therefore conclude that the possibility (3) does occur, in which case $u^{m(u)} = u$ for suitable $m(u) > 1$, since $P(u)$ is a finite field. (What we have actually done in ruling out the possibilities (1) and (2) has been to prove a slight generalization of a theorem of Krasner **(7)**. The proof appearing in his paper could also have been used here, but the argument we have given is of a less complicated nature.)

So far in the proof of Theorem 1 we have shown that if $u$ is any commutator of $D$ then either $u \in Z$ or $u^{m(u)} = u$. Suppose that $u = xy - yx \neq 0 \in Z$. The commutator

$$x = (xu)u^{-1} = [x(xy) - (xy)x]u^{-1} = (xu^{-1})(xy) - (xy)(xu^{-1})$$

does not lie in $Z$. Also the commutator $ux \notin Z$, since

$$(ux)y - y(ux) = u(xy - yx) = u^2 \neq 0.$$

It follows that $x^{n+1} = x$, that is, $x^n = 1$, and $(ux)^{m+1} = u^{m+1}x^{m+1} = ux$, that is, $u^m x^m = 1$, for suitable $m, n > 0$. Therefore

$$1 = (u^m x^m)^n = u^{mn} x^{nm} = u^{mn}, \text{ that is, } u^{mn+1} = u.$$

We thus conclude that for all commutators $u$ of $D$ $u^{m(u)} = u$ where $m(u) > 1$ is an integer. This completes the proof of Theorem 1.

At this point we remark that subrings and homomorphic images of $\gamma$-rings are themselves $\gamma$-rings. Using the Jacobson structure theory, we know that every primitive $\gamma$-ring is either a division ring or else contains a subring which has as a homomorphic image the set $D_2$ of all two by two matrices over some division ring $D$. Since $D_2$ is clearly not a $\gamma$-ring, we have by Theorem 1:

LEMMA 3. *Every primitive $\gamma$-ring is a field.*

The following easy lemma is useful in simplifying our problem:

LEMMA 4. *Suppose a ring $R$ is a subdirect sum of rings $R_\alpha$, each satisfying the polynomial identity $f(\mu_1, \mu_2, \ldots, \mu_m) = 0$ with integer coefficients. Then $R$ also satisfies this identity.*

THEOREM 2. *Every semi-simple $\gamma$-ring $R$ is commutative.*

*Proof.* $R$ is isomorphic to a subdirect sum of primitive rings $R_\alpha$, each of which is a $\gamma$-ring and hence commutative by Lemma 3. Then $R$ is commutative, from Lemma 4, if we choose as our polynomial $f(\mu_1, \mu_2) = \mu_1\mu_2 - \mu_2\mu_1$.

COROLLARY. *If $R$ is any $\gamma$-ring, then every commutator of $R$ lies in the radical $N$.*

**2. The general solution.** Theorem 2 enables us to assume without loss of generality that the (Jacobson) radical $N$ of our $\gamma$-ring is non-trivial. Furthermore, $R$ may be taken to be subdirectly irreducible. Indeed, assuming for the moment that all commutators of subdirectly irreducible $\gamma$-rings are central, any $\gamma$-ring $R$ is a subdirect sum of subdirectly irreducible $\gamma$-rings $R_\alpha$, each of which satisfies the polynomial identity

$$(\mu_1\mu_2 - \mu_2\mu_1)\mu_3 - \mu_3(\mu_1\mu_2 - \mu_2\mu_1) = 0.$$

Then by Lemma 4 $R$ satisfies this same identity, which is precisely the property we wish $R$ to have.

Therefore from now on $R$ will be a $\gamma$-ring with radical $N \neq 0$, centre $Z$, and unique minimal two-sided ideal $S \neq 0$.

LEMMA 5. $S^2 = 0$.*

*Proof.* $S \subset N$ since $N \neq 0$. Let $s \in S$ and $x \in R$. $(sx - xs)^n - (sx - xs) \in S \cap Z$ for some $n = n(s, x) > 1$. If $(sx - xs)^n - (sx - xs) = 0$, then $(sx - xs) = 0$, since $sx - xs \in N$ and no non-zero radical element can be a radical multiple of itself. If

$$u = (sx - xs)^n - (sx - xs) \neq 0,$$

we consider the two-sided ideal $T = uS \subset S$. $T$ must be trivial, for otherwise $T = S$, and $uv = u$ for some $v \in S$ forces a contradiction. Thus

$$ut = [(sx - xs)^{n-1}][(sx - xs)]t - (sx - xs)t = 0$$

for all $t \in S$, from which we get $(sx - xs)t = 0$, since $(sx - xs)t$ is a radical multiple of itself. So far then in our proof we have shown that $(sx - xs)t = 0$ for all $s, t \in S$ and all $x \in R$.

Again let $s \neq 0 \in S$. The right ideal $sS$ is two-sided because

$$(xs)t = (xs - sx)t + s(xt) = s(xt) \in sS$$

for all $t \in S$ and $x \in R$. $sS$ is trivial, for if $sS \neq 0$, then, since it is a two-sided ideal, $sS = S$, and we are faced with the familiar contradiction that $st = s$ for some $t \in S \subset N$. Since the choice of $s$ was arbitrary, $S^2 = 0$.

The next lemma is actually valid for any $\gamma$-ring.

LEMMA 6. *Let $x, y \in R$. Then $(xy - yx)^n c_n - (xy - yx) \in Z$, $n = 1, 2, \ldots,$ where the $c_n$ are suitable polynomials in $xy - yx$.*

*Proof.* We set $w = xy - yx$ and proceed with a proof by induction on $n$. For $n = 1$ we set $m = n(w)$ and choose $c_1 = w^{m-1}$. We now assume the lemma true for $k = n - 1$ and prove it for $k = n$. Indeed, $w^{n-1}c_{n-1} - w \in Z$ by assumption, where $c_{n-1}$ is a polynomial in $w$. We may as well suppose that $m$ is odd, since a similar argument will prevail in case $m$ is even. Then

$$(w^{n-1}c_{n-1} - w)^m = w^n c_n - w^m \in Z,$$

*The proofs of this lemma and the succeeding ones are patterned after those given by Herstein in his papers (**2**; **4**; **5**).

with $c_n$ clearly a polynomial in $w$. Combining this result with the fundamental condition $w^m - w \in Z$, we finally achieve

$$w^n c_n - w = (w^n c_n - w^m) + (w^m - w) \in Z.$$

By choosing a sufficiently large $n$ according to Lemma 6 we are able to state a useful

COROLLARY. *If $xy - yx$ is nilpotent for some $x, y \in R$, then $xy - yx \in Z$.*

LEMMA 7. *Every commutator of $R$ is nilpotent.*

*Proof.* Suppose there exists a commutator $w = xy - yx$ which is not nilpotent. Consider the collection of all ideals of $R$ which enjoy the property that all powers of $w$ fall outside the ideal. The zero ideal is clearly a member of this collection. Partially ordering the collection by set inclusion, we are able to choose by Zorn's Lemma an ideal $U$ which is maximal with respect to the property that $w^n \notin U$ for $n = 1, 2, 3 \ldots$. So if $V$ contains $U$ properly, where $V$ is an ideal of $R$, then $w^{n(V)} \in V$. In other words, for any non-zero ideal $\bar{V}$ of $\bar{R} = R/U$ there exists a natural number $m$, depending on $V$, such that $\bar{w}^m \in \bar{V}$, where $\bar{w}$ denotes the coset $w + U$.

First of all, $\bar{R}$ cannot be subdirectly irreducible. Indeed, suppose that its minimal ideal $\bar{T} \neq 0$. By the corollary to Theorem 2 $w \in N$, which means that $\bar{w}$ is a non-zero element in the radical $\bar{M}$ of $\bar{R}$. Since $\bar{M} \neq 0$, Lemma 5 yields $\bar{T}^2 = 0$. A contradiction is quickly reached when we pick the $m$ such that $\bar{w}^m \in \bar{T}$ and see that $\bar{w}^{2m} = 0$ or $w^{2m} \in U$. Hence we must assume that $\bar{T} = 0$.

Now let $\bar{V}$ be any non-zero ideal of $\bar{R}$. $\bar{w}^m \in \bar{V}$ for sufficiently large $m$. By Lemma 6 $\bar{w}^m\bar{c}_m - \bar{w}$ is in the centre $\bar{Y}$ of $\bar{R}$, where $\bar{c}_m$ is a polynomial in $\bar{w}$. Noting that $\bar{w}^m\bar{c}_m \in \bar{V}$, we see that

$$(\bar{w}^m\bar{c}_m)\bar{r} - \bar{r}(\bar{w}^m\bar{c}_m) = \bar{w}\bar{r} - \bar{r}\bar{w} \in \bar{V}$$

for all $\bar{r} \in \bar{R}$. It follows that for all $\bar{r} \in \bar{R}$ $\bar{w}\bar{r} - \bar{r}\bar{w} = 0$ since the intersection of all the ideals of $\bar{R}$ is 0. In other words, $\bar{w} \in \bar{Y}$.

$\bar{w}\bar{x} = \bar{x}\bar{y}\bar{x} - \bar{y}\bar{x}\bar{x}$ is also a commutator of $\bar{R}$. As we have just shown that $\bar{w} \in \bar{Y}$, $(\bar{w}\bar{x})^k = \bar{w}^k\bar{x}^k$ for all natural numbers $k$. Thus for any non-zero ideal $\bar{V}$ of $\bar{R}$ a sufficiently high power of $\bar{w}\bar{x}$ lies in $V$. Using exactly the same argument as in the proof that $\bar{w} \in \bar{Y}$ but replacing $\bar{w}$ by $\bar{w}\bar{x}$, we can conclude that $\bar{w}\bar{x} \in \bar{Y}$.

Because $\bar{w}$ and $\bar{w}\bar{x}$ are both in $\bar{Y}$, $\bar{w}^2 = \bar{w}(\bar{x}\bar{y} - \bar{y}\bar{x}) = (\bar{w}\bar{x})\bar{y} - \bar{y}(\bar{w}\bar{x}) = 0$, or $w^2 \in U$, a contradiction.

Lemma 7 and the corollary to Lemma 6, together with the remarks made in the opening paragraph of this section yield the

MAIN THEOREM. *If $R$ is a $\gamma$-ring, then every commutator of $R$ lies in the centre of $R$.*

We cannot in general hope to arrive at the sharper conclusion that any $\gamma$-ring is commutative. Indeed, the set of all three by three properly triangular matrices over any field is an example of a non-commutative $\gamma$-ring.

## References


1. N. Jacobson, *Structure theory for algebraic algebras of bounded degree*, Ann. Math., *46* (1945), 695–707.
2. I. N. Herstein, *A generalization of a theorem of Jacobson*, Amer. J. Math., *73* (1951), 756–762.
3. ——— *A generalization of a theorem of Jacobson II*, Abstract, Bull. Amer. Math. Soc., *58* (1952), 383.
4. ——— *A generalization of a theorem of Jacobson III*, Amer. J. Math., *75* (1953), 105–111.
5. ——— *The structure of a certain class of rings*, Amer. J. Math., *75* (1953), 864–871.
6. ——— *A condition for the commutativity of rings*, Can. J. Math., *9* (1957), 583–586.
7. M. Krasner, *The non-existence of certain extensions*, Amer. J. Math., *75* (1953), 112–116.
8. O. F. G. Schilling, *The theory of valuations*, Math. Surveys IV (New York, 1950).
9. H. Weyl, *Algebraic theory of numbers* (Princeton, 1940).


*University of Pennsylvania*
*Yale University*

# ON MATRIX COMMUTATORS

MARVIN MARCUS AND NISAR A. KHAN

**1. Introduction.** Let $A$, $B$, and $X$ be $n$-square matrices over an algebraically closed field $F$ of characteristic 0. Let $[A, B] = AB - BA$ and set $(A, B) = [A, [A, B]]$. Recently several proofs (**1**; **3**; **5**) of the following result have appeared: *if* $\det(AB) \neq 0$ *and* $(A, B) = 0$ *then* $A^{-1}B^{-1}AB - I$ *is nilpotent*. In (**2**) McCoy determined the general form of any $X$ satisfying

$$(1.1) \qquad (A, X) = 0$$

in the case that $A$ has a single elementary divisor corresponding to each eigenvalue, that is, $A$ is *non-derogatory*. In Theorem 1 we determine the structure of any matrix $X$ satisfying (1.1) and also give a formula for the dimension of the linear space of all such $X$ in terms of the degrees of the elementary divisors of $A$. Moreover, we apply our results to obtain a condition that $B$ be a polynomial in $A$. It is a classical result (**6**, p. 150) that *if* $[X, B] = 0$ *whenever* $[A, X] = 0$ *then* $B$ *is a scalar polynomial in* $A$. We prove in Theorem 2 that *if* $(X. B) = 0$ *whenever* $(A, X) = 0$ *then* $B$ *is a scalar polynomial in* $A$.

We also obtain the result that *the dimension of the linear space* $K(A)$ *of all such matrices* $B$ *is precisely the number of distinct eigenvalues of* $A$.

**2. Solutions of** $(A,X) = 0$. Let $A$ have the distinct eigenvalues $\lambda_1, \ldots, \lambda_q$ and let

$$(x - \lambda_i)^{e_{ij}},$$

$j = 1, \ldots, n_i$, $i = 1, \ldots, q$ be the elementary divisors of $A$ with the notation chosen as follows:

For each $i = 1, \ldots, q$

$$e_{i1} > e_{i2} > \ldots > e_{in_i}$$

and

$$(x - \lambda_i)^{e_{ij}}$$

appears with multiplicity $r_{ij}$, $j = 1, \ldots, n_i$.

THEOREM 1. *The number of linearly independent solutions of*

$$(1.1) \qquad (A, X) = 0$$

*is*

Received February 9, 1959; in revised form July 22, 1959. The work of the first author was supported by U.S. National Science Foundation Grant N.S.F. G-5416. The second author is a Post-doctorate Fellow of the National Research Council of Canada.

$$\sum_{i=1}^{q}\left\{\sum_{j=1}^{n_i}(2e_{ij}-1)r_{ij}^2+4\sum_{j<k}^{n_i} r_{ij}r_{ik}e_{ik}\right\}. \tag{2.1}$$

It is clear that we may assume $A$ to be in Jordan canonical form $J$. Set

$$p_i = \sum_{j=1}^{n_i} r_{ij}e_{ij},$$

the algebraic multiplicity of $\lambda_i$.

We write

$$A = \sum_{i=1}^{q}{}^{\cdot} (\lambda_i I_{p_i} + V_i) \tag{2.2}$$

where $I_{p_i}$ is a $p_i$-square identity matrix, $V_i$ is a $p_i$-square matrix with only 1 and 0 in the superdiagonal, all other elements 0, and $\sum^{\cdot}$ indicates direct sum.

We also write

$$V_i = \sum_{j=1}^{n_i}{}^{\cdot} U_{ij},\ J_i = \lambda_i I_{p_i} + \sum_{j=1}^{n_i}{}^{\cdot} U_{ij} \tag{2.3}$$

and hence

$$A = \sum_{i=1}^{q}{}^{\cdot} \sum_{j=1}^{n_i}{}^{\cdot} (\lambda_i I e_{ij} + U_{ij}). \tag{2.4}$$

where $U_{ij}$ is the direct sum of the $e_{ij}$-square auxiliary unit matrix repeated $r_{ij}$ times, $j = 1, \ldots, n_i$. We partition $X$ conformally with the partitioning of $A$ indicated in (2.4). Now consider a block of $X$, call it $C$, that corresponds to $\lambda_i$ and $\lambda_j$ for $i \neq j$. A result obtained in both **(2)** and **(4)** is

LEMMA 1. $C = 0$.

From Lemma 1 we conclude that

$$X = \sum_{i=1}^{q}{}^{\cdot} X_i$$

and $X_i$ is a $p_i$-square matrix. To determine the structure of $X_i$ we may obviously confine our attention to the case in which $A$ has a single eigenvalue with several elementary divisors.

LEMMA 2. *Let $A$ be an $n$-square matrix with the single eigenvalue $\lambda$ and let $(x-\lambda)^{\nu_i}$ be an elementary divisor of $A$ of multiplicity $r_i$, $i = 1, \ldots, t$, $\nu_1 > \ldots > \nu_t$, $\sum_{i=1}^{t} r_i \nu_i = n$. Then the most general matrix $X$ satisfying* (1.1) *contains*

$$\sum_{i=1}^{t} (2\nu_i - 1)r_i^2 + 4\sum_{i<j}^{t} r_i r_j \nu_j$$

*arbitrary parameters.*

*Proof*. Since (1.1) holds for $A - \lambda I$ if and only if it holds for $A$ we may assume $\lambda = 0$ without loss of generality. Thus we assume

$$A = \sum_{i=1}^{s}{}^{\cdot} \sum_{j=1}^{r_i}{}^{\cdot} U_i \tag{2.5}$$

where $U_i$ is a $\nu_i$-square matrix with 1 along the superdiagonal and 0 elsewhere. If $\nu_i = 1$ then $U_i$ is the 1-square 0 matrix. We partition $X$ conformally with $A$ in (2.5):

$$X = (X_{ij}),$$

and observe from (1.1) that

$$U_i^2 X_{ij} + X_{ij} U_j^2 - 2U_i X_{ij} U_j = 0. \tag{2.6}$$

For the sake of simplicity of notation, we take $U_i = U$ as $m$-square, $U_j = V$ as $n$-square, and $X_{ij} = C = (c_{ij})$ as an $m \times n$ matrix. There are three essentially distinct cases to consider:

$$\text{(i) } m = n, \qquad \text{(ii) } m > n, \qquad \text{(iii) } m < n.$$

The case (i), $m = n$, is done in **(2)** and **(4)** and in this case $c_{ij} = 0$ $i > j$, and the elements of each diagonal parallel (or equal) to the main diagonal are in arithmetic progression. Hence the number of arbitrary parameters in $C$ in case (i) is $2(n-1) + 1 = 2n - 1$. The case $n \leqslant 2$ is not considered in **(2)** but it is trivial to see that the number of parameters there is also $2n - 1$.

Case (ii): $m > n$. We have from (2.6)

$$U^2 C + CV^2 - 2UCV = 0. \tag{2.7}$$

We assume in what follows that $m \geqslant 3$. The case $m = 2$, $n = 1$ will be disposed of later. Let $\epsilon_i$ be the unit column $n$-vector with 1 in position $i$, $i = 1, \ldots, n$. We evaluate the transform of $\epsilon_i$ by the left side of (2.7) to obtain

$$U^2 c_i + c_{i-2} = 2Uc_{i-1}, \tag{2.8}$$

where $c_i$ denotes the $i$th column of $C$, $i = 1, \ldots, n$. Co-ordinatewise (2.8) becomes for $i = 1, \ldots, n$

$$(2.9)_i \quad [c_{3i}, c_{4i}, \ldots, c_{mi}, 0, 0] + [c_{1,i-2}, c_{2,i-2}, \ldots, c_{m-2,i-2}, c_{m-1,i-2}, c_{m,i-2}] \\ = 2[c_{2,i-1}, c_{3,i-1}, \ldots, c_{m-1,i-1}, c_{m,i-1}, 0].$$

We show first that $c_{ij} = 0$ for $i > j$, $i \geqslant 3$. We compute the $(i + s)$ coordinate of $(2.9)_i$ where $s$ is one of the integers $0, \ldots, m - i$;

$$c_{i+s+2,i} + c_{i+s,i-2} - 2c_{i+s+1,i-1} = 0, \quad i = 1, \ldots, m - (s+2). \tag{2.10}$$

We first note that from (2.8) for $i = 1$ we have

$$U^2 c_1 = 0$$

and hence

$$c_{31} = c_{41} = \ldots = c_{m1} = 0. \tag{2.11}$$

Consider the system (2.10) for $i = 2, 3, \ldots, m - (s+2)$ in succession and obtain

$$\begin{aligned} &c_{s+4,2} = 2c_{s+3,1} = 0 \qquad \text{(by 2.11)} \\ &c_{s+5,3} = -c_{s+3,1} + 2c_{s+4,2} = 0 \\ &\cdots \\ &c_{m,m-(s+2)} = -c_{m-2,m-s-4} + 2c_{m-1,m-s-3} = 0. \end{aligned}$$

Thus $c_{ij} = 0$ for $i > j$, $i \geqslant 3$. We now consider in succession the $(i-1)$ co-ordinate of $(2.9)_i$ for $i = 2, \ldots, n$ (since $n + 1 \leqslant m$), to obtain

$$c_{i+1,i} + c_{i-1,i-2} - 2c_{i,i-1} = 0.$$

Setting $i$ successively equal to $2, \ldots, n$ we have

$$c_{32} = 2c_{21},\ c_{43} = 3c_{21}, \ldots, c_{n+1,n} = nc_{21}.$$

Hence there is only one arbitrary parameter $c_{21}$ in this diagonal of $C$. We next consider the elements $c_{ij}$ for $i \leqslant j$. We compute the $r$th co-ordinate of $(2.9)_{r+s}$, $r = 1, \ldots, n-s$, where $s$ is one of the integers $2, \ldots, n-1$, to obtain

$$\begin{aligned} &c_{3,s+1} + c_{1,s-1} = 2c_{2,s} \\ &c_{4,s+2} + c_{2,s} = 2c_{3,s+1} \\ &\cdots \\ &c_{n-s+2,n} + c_{n-s,n-2} = 2c_{n-s+1,n-1}. \end{aligned}$$

Hence

$$c_{1,s-1},\quad c_{2,s},\quad c_{3,s+1}, \ldots, c_{n-s+2,n}$$

are in arithmetic progression. Thus in each diagonal

$$d_{s-1} = c_{1,s-1}, \ldots, c_{n-s+2,n} \quad s = 2, \ldots, n-1$$

there are two arbitrary parameters. In the diagonals $d_{n-1}$ and $d_n$ we accumulate three more arbitrary parameters in $C$, $c_{1,n-1}$, $c_{1n}$, $c_{2n}$. Hence the total number of arbitrary parameters in $C$ for $m > n$ is

$$3 + 2(n-2) + 1 = 2n.$$

We compute easily that for $m = 2$, $n = 1$, $C$ involves $2n = 2$ arbitrary parameters as well.

Case (iii): $n > m$. We reduce this to case (ii) as follows:

Let $P_k$ denote the $k$-square permutation matrix with 1 in each of the positions $(k-j, j+1)$, $j = 0, \ldots, k-1$. Taking the transpose of (2.7) we have

$$C'(U')^2 + (V')^2C' - 2V'C'U' = 0. \tag{2.12}$$

Now observe that

$$\begin{aligned} U' &= P_m U P_m, \\ V' &= P_n V P_n, \end{aligned}$$

and substituting in (2.12) using $P_m^2 = I_m$, $P_n^2 = I_n$ we have

$$C'P_mU^2P_m + P_nV^2P_nC' - 2P_nVP_nC'P_mUP_m = 0.$$

Now pre-multiplying by $P_n$ and post-multiplying by $P_m$ we have

$$(P_nC'P_m)U^2 + V^2(P_nC'P_m) - 2V(P_nC'P_m)U = 0.$$

Now $P_nC'P_m$ is $n \times m$, the situation of case (ii). Hence $P_nC'P_m$ has $2m$ arbitrary parameters and $C$ has $2m$ arbitrary parameters.

Returning to the statement of Lemma 2, we conclude from case (i) that any block in the partitioning of $X$ corresponding to equal $U_i$'s contains $2\nu_i - 1$ arbitrary parameters and there are $r_i^2$ such blocks for each $i$. Also from (ii) and (iii) any block in $X$ corresponding to $U_i$ and $U_j$, $i < j$, contains $2\nu_j$ arbitrary parameters (since for $i < j$, $\nu_i > \nu_j$). Hence the total number of arbitrary parameters in $X$ is

$$\sum_{i=1}^{t} (2\nu_i - 1)r_i^2 + 4 \sum_{i<j}^{t} r_ir_j\nu_j.$$

We return now to the proof of Theorem 1. By Lemma 1 we need only add the total number of parameters of the $q$ main diagonal blocks of $X$ corresponding to each $\lambda_i$. By Lemma 2, this number for a fixed $\lambda_i$ is

$$\sum_{j=1}^{n_i} (2e_{ij} - 1)r_{ij}^2 + 4 \sum_{j<k}^{n_i} r_{ij}r_{ik}e_{ik}.$$

Summing this for $i = 1, \ldots, q$ we obtain the formula (2.1) and the proof is complete.

**3. The space $K(A)$.** Let $P$ be a non-singular matrix satisfying

$$P^{-1}AP = J \tag{3.1}$$

and let $Y = P^{-1}XP$ and $C = P^{-1}BP$. Then $(P^{-1}AP, P^{-1}BP) = P^{-1}(A, B)P$ implies that $B \in K(A)$ if and only if $C \in K(J)$. As indicated earlier, if $(J, Y) = 0$, then

$$Y = \sum_{s=1}^{q}{}^{\cdot}\, Y_s \tag{3.2}$$

and $Y_s$ is $p_s$-square, $s = 1, \ldots, q$. If

$$Y_s = (Y_{ij}),\ i, j = 1, \ldots, m_s,\ m_s = \sum_{j=1}^{n_s} r_{sj} \tag{3.3}$$

indicates a partitioning of $Y_s$ conformally with the partitioning of $J_s$ in (2.3) then we have seen in the proof of Theorem 1 that a block in (3.3) is an $e_{si} \times e_{sj}$ rectangular matrix with the following structure:

**1.** $e_{si} > e_{sj}$.

(i) each diagonal, except the element in the upper right corner, parallel or equal to the diagonal starting from the upper left corner involves two arbitrary parameters, and the elements in each of these diagonals are in arithmetical progression;

(ii) there is one non-zero diagonal immediately below the diagonal starting from the upper left corner, containing one parameter only. The elements are of the form $a, 2a, 3a, \ldots, e_{sj}a$ for an arbitrary $a \in F$;

(iii) all other elements are zero.

**2.** $e_{si} < e_{sj}$.

(i) the diagonal $d$ ending in the lower right corner and those above it each involve two arbitrary parameters and the elements are in arithmetical progression, with the exception of the upper right corner element which is arbitrary;

(ii) the diagonal immediately below $d$ contains one parameter and the elements are of the form $e_{si}a, \ldots, 3a, 2a, a$ for an arbitrary $a \in F$;

(iii) all other elements are zero.

**3.** $e_{si} = e_{sj}$.

The block is upper triangular. Each diagonal involves two arbitrary parameters and the elements are in arithmetical progression with the exception of the upper right corner element which is arbitrary.

Let $L(J)$ be the linear space of all $Y$ satisfying $(J, Y) = 0$.

LEMMA 3. $(Y, C) = 0$ *for each* $Y \in L(J)$ *if and only if*

$$C = \sum_{j=1}^{q} c_j I_{p_j} \tag{3.4}$$

*where* $c_j \in F$, $j = 1, 2, \ldots, q$.

*Proof*. The sufficiency of (3.4) is clear. By the above description of $L(J)$,

$$\sum_{j=1}^{q} x_j I_{p_j} \in L(J)$$

for any $x_j \in F$.

Hence $(Y, C) = 0$ implies

$$C = \sum_{s=1}^{q} C_s,$$

$C_s$ is $p_s$-square. Now $(Y, C) = 0$ implies that $(Y_s, C_s) = 0$, $s = 1, \ldots, q$. We may choose $Y \in L(J)$ with

$$Y_s = \sum_{j=1}^{n_s} \sum_{\alpha=1}^{r_{sj}} x_{j\alpha} I_{e_{sj}}$$

for arbitrary $x_{j\alpha} \in F$ and conclude that

$$C_s = \sum_{j=1}^{n_s} C_{sj},$$

where $C_{sj}$ is a direct sum of $r_{sj}$ $e_{sj}$-square matrices $j = 1, \ldots, n_s$. Let $M_{sj}$ be any one of the $e_{sj}$-square blocks whose direct sum comprises $C_{sj}$.

We next show that $C_s$ is a scalar multiple of the identity by noting first that $(Y_s, C_s) = 0$ implies that $(D_{sj}, M_{sj}) = 0$ where $D_{sj}$ is an $e_{sj}$-square diagonal matrix with diagonal elements (in arithmetical progression) along the main diagonal. Hence $M_{sj}$ is diagonal. Let

$$M_{sj} = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{e_{sj}}).$$

Now we may choose $Y_s$ such that the equation $(E_{sj}, M_{sj}) = 0$ holds, where $E_{sj}$ is $e_{sj}$-square and

$$E_{sj} = \begin{bmatrix} 0 & x & 0 & \cdots & 0 \\ 0 & 0 & x+y & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & x+(e_{sj}-2)y \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Now $(E_{sj}, M_{sj}) = 0$ is equivalent (for the case $e_{sj} > 3$, the case $e_{sj} < 2$ is trivial) to

$$E_{sj}^2 M_{sj} + M_{sj} E_{sj}^2 = 2E_{sj} M_{sj} E_{sj}.$$

Elementwise we have

$$(x + (t-3)y)\,(x + (t-2)y)\,(\alpha_p + \alpha_{p-2} - 2\alpha_{p-1}) = 0$$

for $t = 3, \ldots, e_{sj}$ and for arbitrary $x, y$. Hence we conclude that

$$\alpha_1, \alpha_2, \ldots, \alpha_{e_{sj}}$$

are in arithmetical progression and thus we may write

$$M_{sj} = \operatorname{diag}(\alpha, \alpha+\beta, \ldots, \alpha + (e_{sj}-1)\beta). \tag{3.5}$$

We next show that $\beta = 0$. To this end we choose $Y_s$ such that $(Y_s, C_s) = 0$ implies the following:

$$(F_{sj}, M_{sj}) = 0, \tag{3.6}$$

where

$$F_{sj} = \begin{bmatrix} 1 & 1 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & e_{sj} \end{bmatrix}.$$

From (3.5) we obtain by computing the (1, 2) element of (3.6) that $\beta = 0$. Hence

$$C_s = \sum_{j=1}^{m_s} C_{sj}.$$

where

$$C_{sj} = \sum_{\alpha=1}^{r_{sj}} x_{j\alpha} I_{e_{sj}},$$

We next show that all $x_{j\alpha}$ are equal. Let

$$y_1 = x_{11},\ y_2 = x_{12}, \ldots, y_{r_{s1}} = x_{1r_{s1}},\ y_{r_{s1}+1} = x_{21}, \ldots, y_{m_s} = x_{n_s r_{sn_s}}.$$

Then from $(Y_s, C_s) = 0$, it follows that

$$\sum_{j=1}^{m_s} y_v Y_{uj} Y_{jv} + \sum_{j=1}^{m_s} y_u Y_{uj} Y_{jv} = 2 \sum_{j=1}^{m_s} y_j Y_{uj} Y_{jv}, \quad u, v = 1, 2, \ldots, m_s.$$

We have, by computing the block in the upper left corner (that is, the one conformal with $Y_{11}$),

$$y_1 \sum_{j=1}^{m_s} Y_{1j} Y_{j1} = \sum_{j=1}^{m_s} y_j Y_{1j} Y_{j1}.$$

$$\sum_{j=2}^{m_s} (y_1 - y_j) Y_{1j} Y_{j1} = 0. \tag{3.7}$$

For a fixed $t$, $1 < t < m_s$, choose the $e_{s1} \times e_{st}$ matrix (recalling that $e_{s1} > e_{st}$)

$$Y_{1t} = \sum_{j=1}^{e_{st}} G_{jj},$$

where $G_{jj}$ is an $e_{s1} \times e_{st}$ matrix with 1 in the $(j, j)$ position and zeros elsewhere. Also choose

$$Y_{t1} = \sum_{j=1}^{e_{st}} H_{jj},$$

where $H_{jj}$ is $e_{st} \times e_{s1}$ with 1 in the $(j, j)$ position and zeros elsewhere, and let $Y_{1j} = 0$ for $j \neq t$. Then (3.7) becomes

$$(y_1 - y_t) I_{e_{s1}} = 0.$$

Hence $y_1 = y_t$, $t = 2, 3, \ldots, m_s$ and $C$ has the form indicated in (3.4).

THEOREM 2. *If $(X, B) = 0$ for any $X$ satisfying $(A, X) = 0$, then $B$ is a scalar polynominal in $A$.*

*Moreover the dimension of the linear space $K(A)$ of all such $B$ is given by*

$$\dim K(A) = q,$$

*where $q$ is the number of distinct eigenvalues of $A$.*

*Proof.* We have seen that $B \in K(A)$ if and only if $C = P^{-1}BP \in K(J)$ where $P^{-1}AP = J$ is the Jordan canonical form of $A$.

Moreover, by Lemma 3

$$C = \sum_{j=1}^{q} c_j I_{p_j},$$

where $c_j \in F$ are arbitrary. This implies immediately that

$$\dim K(A) = q.$$

Now

$$B = PCP^{-1} = P\left(\sum_{j=1}^{q} c_j I_{p_j}\right)P^{-1}.$$

Let $0_p$ be the $p$-square matrix of zeros and let

$$E_j = P(0_{u_j} \dotplus I_{p_j} \dotplus 0_{v_j})P^{-1},$$

where

$$u_j = \sum_{i=1}^{j-1} p_i, v_j = n - \sum_{i=1}^{j} p_i, \ j = 1, 2, \ldots, q.$$

Then the $E_j$ are the principal idempotents of $A$ corresponding to the $\lambda_j$ respectively and each $E_j$ is a scalar polynomial $f_j(A)$ in $A$ (**7**, p. 29).

Hence

$$B = \sum_{j=1}^{q} c_j E_j = \sum_{j=1}^{q} c_j f_j(A)$$
$$= f(A)$$

where

$$f(x) = \sum_{j=1}^{q} c_j f_j(x).$$

### References


1. I. N. Herstein, *On a theorem of Putnam and Wintner*, Proc. Amer. Math. Soc., *9* (1958), 363–364.
2. N. H. McCoy, *On quasi-commutative matrices*, Trans. Amer. Math. Soc., *36* (1934), 327–340.
3. C. R. Putnam and A. Wintner, *On the spectra of group commutators*, Proc. Amer. Math. Soc., *9* (1958), 360–362.
4. W. E. Roth, *On k-commutative matrices*, Trans. Amer. Math. Soc., *39* (1936), 483–495.
5. O. Taussky, *A note on the group commutator of A and A**, J. Wash. Acad. Sci., *48* (1958), 305.
6. H. W. Turnbull and A. C. Aitken, *An introduction to the theory of canonical matrices* (London, 1932).
7. J. H. M. Wedderburn, *Lectures on matrices*, Amer. Math. Soc. Colloq. Publ. *17* (New York, 1934).



*The University of British Columbia*
*and*
*Muslim University, Aligarh*

# ON NULL-RECURRENT MARKOV CHAINS

JOHN LAMPERTI

**1. Introduction.** Throughout this paper, the symbol $P = [P_{ij}]$ will represent the transition probability matrix of an irreducible, null-recurrent Markov process in discrete time. Explanation of this terminology and basic facts about such chains may be found in **(6**, ch. 15**)**. It is known **(3)** that for each such matrix $P$ there is a unique (except for a positive scalar multiple) positive vector $Q = \{q_i\}$ such that $QP = Q$, or

(1) $$q_j = \sum_i q_i P_{ij};$$

this vector is often called the "invariant measure" of the Markov chain.

The first problem to be considered in this paper is that of determining for which vectors $U^{(0)} = \{\mu_i^{(0)}\}$ the vectors $U^{(n)}$ converge, or are summable, to the invariant measure $Q$, where $U^{(n)} = U^{(0)}P^n$ has components

(2) $$\mu_j^{(n)} = \sum_i \mu_i^{(n-1)} P_{ij} = \sum_i \mu_i^{(0)} P_{ij}^{(n)}.$$

In § 2, this problem is attacked for general $P$. The main result is a negative one, and shows how to form $U^{(0)}$ for which $U^{(n)}$ will not be (termwise) Abel summable. As this negative result shows, the operators formed from $P$ do not obey the mean ergodic theorem. (It is interesting to contrast this situation with the case when $P$ is the stochastic matrix of an *ergodic* Markov chain **(6)**.) However, in § 3 more inclusive positive results are found for two special classes of matrices $P$.

The invariant measure may be used to form a stationary process as follows: let $N_i^0$ be independent Poisson random variables with respective means $q_i$, and suppose that at time 0, $N_i^0$ particles* are placed in state $i$ of the Markov chain. Suppose that each particle thereafter moves according to the law of the chain independently of the others, and let $N_i^n$ be the number of particles in state $i$ at time $n$. Then for each $n$, the random variables $N_i^n$ are independent, Poisson, and have means $q_i$. These facts are due to Derman **(4**, Theorem 2**)**. It is then natural to ask if there are not non-stationary processes associated with $P$ which converge to this stationary process as a limit, and in fact Derman has already done so in **(4)**. The vectors $A_n$ with components $N_i^n$ form a Markov process which has an "invariant measure"; and the general

Received January 19, 1959. This work was supported by the U.S. Office of Naval Research.

*We shall speak of "moving particles" throughout without further apology. More exact statements may easily be supplied (as been done in **(4)**) at some cost in intuitive appeal.

theory of such processes together with other arguments, yields a variety of results. Here we will take a slightly different point of view.

Actually, the first problem described above is closely connected with the second in the following way: let $N_i^0$ be independent, non-negative integer-valued random variables with means $\mu_i^{(0)}$, and as in the construction of Derman's stationary process put $N_i^0$ particles into state $i$ at time 0 and let the particles move independently. Then it is not hard to see that $N_i^n$, the number of particles in state $i$ at time $n$, has mean $\mu_i^{(n)}$. Therefore the termwise convergence of $U^{(n)}$ is a necessary condition for the existence of Poisson limiting distributions for the $N_i^n$. In § 4, we give a sufficient condition that these limiting distributions exist and are Poisson; the condition roughly is that $U^{(n)}$ converge termwise and the variances of the $N_i^0$ are not too large. This theorem is closely related to Theorem 6 of **(4)**, and the proof is similar; however, our result, with the aid of the material of §§ 2 and 3, does apply to certain cases where the hypotheses of **(4)** are not satisfied. This sort of theorem has also been discussed by several other authors for the case of spatially homogeneous (sums of random variables) processes.

In § 5 a different sort of convergence is considered. Instead of putting all the particles into the various states of the Markov chain at time 0, they can be introduced continually into some fixed state and allowed to diffuse away from it. It is shown that this can always be done in such a way as to obtain convergence to Derman's stationary process. This sort of process was suggested to the author by F. Spitzer.

Finally, we observe that many of our results have analogues in the case of continuous time-parameter Markov chains. With the aid of recent work on such chains, mainly that of K. L. Chung, it appears that proofs quite similar to those in the discrete case can be given. These ideas will not be carried out in this paper.

**2. The convergence of $U^{(n)}$.** The multiplications $U^{(0)}P^n$ will be well defined provided that

$$|\mu_i^{(0)}| < Mq_i, \tag{3}$$

and this condition will be assumed hereafter. It is also assumed that the $\mu_i^{(0)}$ are real.

THEOREM 1. *Suppose that for each $i$ the sequence $\mu_i^{(n)}$ is Abel summable; call the limits $\mu_i$. Then there is a constant $\alpha$ such that $\mu_i = \alpha q_i$ for all $i$.*

*Proof.* Without loss of generality we can assume that $\mu_i^{(0)} > 0$. For in any case, there is an $M$ such that

$$\nu_i^{(0)} = \mu_i^{(0)} + Mq_i > 0,$$

and $\nu_i^{(n)}$ is summable to $\mu_i + Mq_i$; if these numbers are multiples of the $q_i$ by a constant, so are the $\mu_i$.

Now proceeding formally we have

$$\sum_i \mu_i P_{ij} = \sum_i P_{ij} \lim_{x\to 1-} \left\{(1-x)\sum_n \mu_i^{(n)} x^n\right\} = \lim_{x\to 1-} (1-x)\sum_i P_{ij} \sum_n \mu_i^{(n)} x^n$$
$$= \lim_{x\to 1-} (1-x) \sum_n x^n \sum_i \mu_i^{(n)} P_{ij} = \lim_{x\to 1-} \frac{1-x}{x} \sum_n \mu_j^{(n+1)} x^{n+1} = \mu_j.$$

Actually the exchanges of limits are justified; the first one since

$$0 < (1-x) \sum_n \mu_i^{(n)} x^n < Mq_i \qquad \text{for all } x \in (0,1),$$

so that the sum over $i$ is uniformly convergent, and the second since the terms summed are non-negative. Hence, $\mu_i$ are a solution of (1), and $\mu_i > 0$ since we assumed $\mu_i^{(0)} > 0$ for all $i$. The uniqueness of such solutions completes the proof.

We shall use $(m)$ to denote the Banach space of bounded sequences of real numbers (sup norm) and (c) for the subspace of convergent sequences. This notation and other facts about Banach spaces used below may be found in **(1)**.

THEOREM 2. *Let $\{x_i\} \in (c)$, and let $\mu_i^{(0)} = x_i q_i$. Then for all $i$,*

$$\lim_n \mu_i^{(n)} = q_i \lim \{x_j\}. \tag{4}$$

*Proof.* We can assume that $x_i \to 0$, since if the limit is $\alpha$ we consider $\mu_i^{(0)} - \alpha q_i$ instead of $\mu_i^{(0)}$. If $\epsilon > 0$, by the hypothesis we can put

$$\mu_i^{(0)} = \nu_i^{(0)} + \omega_i^{(0)},$$

where $|\nu_i^{(0)}| < \frac{1}{2} q_i \epsilon$ and only a finite number of $\omega_i^{(0)}$ are different from 0. Now for each $j$

$$\omega_j^{(n)} = \sum_i \omega_i^{(0)} P_{ij}^{(n)} \to 0,$$

since $P_{ij}^{(n)} \to 0$ for each $i, j$ **(6)**. Also, it is easy to see from (1) that

$$|\nu_j^{(n)}| = \left|\sum_i \nu_i^{(0)} P_{ij}^{(n)}\right| < \tfrac{1}{2}\epsilon q_j.$$

Hence for large $n$, $|\mu_i^{(n)}| < \epsilon q_i$ and so $\mu_i^{(n)} \to 0$ for every $i$. This and the remark above prove (4).

The theorem just proved seems far from what one might hope for, but it does provide a class of sequences which, *in any order*, when multiplied by the invariant measure yield a vector $U^{(0)}$ for which $U^{(n)}$ is convergent. Actually, the convergent sequences are the only ones with this property:

THEOREM 3. *For each $P$ and each sequence $\{x_i\} \in (m)$ but which is not convergent, there is a permutation $\pi$ of the positive integers such that if $U^{(0)}$ is formed using the rearrangement of $\{x_i\}$, that is,*

$$\mu_i^{(0)} = x_{\pi i} q_i,$$

*then for some $i$, $\mu_i^{(n)}$ fails to be Abel summable.*

The proof rests on two lemmas, which may be of independent interest. The first is a much less precise version of the theorem.

LEMMA 1. *For each $P$, there exists a sequence $\{x_i\} \in (m)$ such that if $\mu_i^{(0)} = x_i q_i$ then for some $i$, $\mu_i^{(n)}$ is not Abel summable.*

*Proof.* Let $l(q)$ be the Banach space of sequences $\{y_i\}$ such that

$$||y|| = \sum_i |y_i|q_i < \infty.$$

With respect to the inner product

$$(y, x) = \sum_i y_i x_i q_i,$$

$(m)$ is the conjugate space of $l(q)$. We can use the matrix $P$ to define an operator $T$ of norm one on $l(q)$:

$$\{Ty\}_i = \sum_j P_{ij} y_j.$$

It is easily verified that the operator $T^*$ on $(m)$ defined by

$$\{T^*x\}_j = \frac{1}{q_j} \sum_i x_i q_i P_{ij}$$

is the adjoint of $T$.

Let $\delta^k$ stand for the sequence whose $k$th term is one and the rest zero. Then for $|z| < 1$,

$$\left((1 - z) \sum_n z^n T^n \delta^j, \delta^i\right) = q_i(1 - z) \sum_n P_{ij}^{(n)} z^n$$

which has limit 0 as $z \to 1 -$. This means that for fixed $j$ if the vectors $(1 - z) \sum_n z^n T^n \delta^j$ have a weak limit as $z \to 1 -$, that limit must be zero. But for any $z < 1$,

$$\left((1 - z) \sum_n z^n T^n \delta^j, 1\right) = q_j.$$

Therefore $(1 - z) \sum_n z^n T^n \delta^j$ does not converge weakly to zero, and hence has no weak limit. But the space $l(q)$ is weakly complete; we conclude that there exists a linear functional (that is, a sequence $\{x_i\} \in (m)$) such that

$$\left((1 - z) \sum_n z^n T^n \delta^j, x\right) = (1 - z) \sum_n z^n (\delta^j, T^{*n} x)$$
$$= (1 - z) \sum_n z^n \sum_i x_i q_i P_{ij}^{(n)} = (1 - z) \sum_n z^n \mu_j^{(n)}$$

does not have a limit as $z \to 1 -$. This is the assertion of the lemma.

*Remark.* We have actually proved that for each $j$, there is an $x \in (m)$ such that if $\mu_i^{(0)} = x_i q_i$, then the $\mu_j^{(n)}$ are not summable. It is not hard to add that there is an $x$ with non-negative components such that for no $j$ does $\mu_j^{(n)}$ form an Abel summable sequence; this is not needed in the proof of Theorem 3.

LEMMA 2. *In the class of closed subspaces of* $(m)$ *which are invariant under all reorderings,* $(c)$ *is maximal.*

*Proof.** Let $(d)$ be a closed subspace of $(m)$ containing $(c)$ and such that $\{x_i\} \in (d)$ implies $\{x_{\pi i}\} \in (d)$ for any permutation $\pi$ of the integers. Suppose that $(d)$ contains a sequence consisting only of 1's and 0's and with an infinite number of each. Then $(d)$ contains all sequences containing only 1's and 0's.

A sequence will be called "simple" if it contains only finitely many different numbers. Under our assumptions, all simple sequences belong to $(d)$, since such a sequence is a finite linear combination of sequences consisting of 1's and 0's. But any bounded sequence can be uniformly approximated by simple ones, so $(d)$ must equal $(m)$.

Now suppose instead that $(d)$ contains some sequence which is not convergent, say $\{y_i\}$. Then there must be two convergent subsequences, say $\{y_{n(i)}\}$ and $\{y_{m(i)}\}$, with limits $\alpha$ and $\beta$ respectively, $\alpha \neq \beta \neq 0$. By adding two suitable convergent sequences and multiplying by a constant, we can see that $\{z_i\} \in (d)$, where

$$z_{n(i)} = 1,\, z_{m(i)} = 0, \qquad \text{and} \qquad z_j = \frac{y_j - \beta}{\alpha - \beta}$$

for other values of $j$. $(d)$ then also contains $\{w_i\}$ where

$$w_{n(2i)} = 1,\ w_{n(2i-1)} = 0 = w_{m(i)}, \quad \text{and} \quad w_j = z_j$$

otherwise, since this sequence was obtained from $\{z_i\}$ by a rearrangement. Subtracting, we obtain a non-convergent sequence of 1's and 0's only which belongs to $(d)$, and by the argument above we conclude that $(d) = (m)$, which completes the proof.

*Proof of Theorem* 3. The quantities $\mu_i^{(n)}$ are now defined by (2) using the matrix $P$ under consideration. Let $(e)$ be the class of all sequences $\{x_i\} \in (m)$ such that if $\mu_i^{(0)} = x_{\pi i}q_i$, then $\mu_j^{(n)}$ is Abel-summable for each $j$ and for each permutation $\pi$ of the integers. By definition, $(e)$ is invariant under reorderings; by Theorem 2 $(e) \supset (c)$. It is not hard to see that $(e)$ is a closed subspace of $(m)$, but from Lemma 1 we know that $(e) \neq (m)$. Therefore by Lemma 2, $(e) = (c)$, which proves the theorem.

**3. Special classes of $P$.** In the previous section, dealing with arbitrary $P$ having no intrinsic ordering of the states, we considered sequences giving rise in any ordering to convergent $\mu_j^{(n)}$. Here we shall look at certain Markov chains in which there is a natural order for the states. First consider a Markov process consisting of sums of independent, identically-distributed random variables $X_n$ taking integer values, and let $P$ be the matrix of transition probabilities. Then $q_j = 1$ for all $j$ is a solution of (1); the sequences $\mu_j^{(0)}$ satisfying (3) are just those in $(m)$. Let

*A discussion with Halsey Royden was very helpful in proving this lemma.

$$p_j = \Pr(X_n = j) \quad \text{and} \quad \phi(x) = \sum_{n=-\infty}^{\infty} e^{inx} p_n.$$

(In this section, the letter $i$ will not be used for an index.)

THEOREM 4. *If*

$$(5) \qquad \mu_j^{(0)} = \nu_j^{(0)} + \int_{-\pi}^{\pi} e^{ijx} dF(x),$$

*where* $\nu_j^{(0)} \to 0$ *as* $j \to \pm \infty$ *and* $F(x)$ *is of bounded variation, then* $\mu_k^{(n)}$ *is Cesàro summable for each* $k$. *If* $\gcd\{j : p_j = 0\} = 1$, *the sequences* $\mu_k^{(n)}$ *are convergent.*

*Proof.* In view of Theorem 2, we can assume the $\nu_j^{(0)}$ are zero. Then by (5),

$$\mu_k^{(n)} = \sum_j \mu_j^{(0)} P_{jk}^{(n)} = \int_{-\pi}^{\pi} \sum_j P_{jk}^{(n)} e^{ijx} dF(x).$$

But

$$P_{jk}^{(n)} = \Pr(X_1 + \ldots + X_n = k - j),$$

so that

$$\sum_j P_{jk}^{(n)} e^{i(k-j)x} = \phi^n(x).$$

Hence

$$(6) \qquad \mu_k^{(n)} = \int_{-\pi}^{\pi} e^{ikx} \phi^{-n}(x) dF(x).$$

If $\gcd\{j : p_j = 0\} = 1$, $x = 0$ is the only point in $[-\pi, \pi]$ at which $|\phi(x)| = 1$, and so $\mu_k^{(n)}$ converges to the jump of $F(x)$ at 0. If the gcd $= d$, say, then $|\phi(x)| = 1$ only when $e^{ix}$ is a $d$th root of unity, and Cesàro convergence follows from (6).

Another type of Markov chain with intrinsic ordering of the states is a random walk. Karlin and McGregor have shown **(7)** that for every random walk on the non-negative integers there is a non-decreasing function $\psi(x)$ such that

$$(7) \qquad P_{jk}^{(n)} = q_k \int_{-1}^{1} x^n Q_j(x) Q_k(x) d\psi(x),$$

where as usual $q_j$ are a solution of (1) and the $Q_j(x)$ are the orthogonal polynomials of the measure $d\psi(x)$, normalized so that $Q_j(1) = 1$.

THEOREM 5. *If, in the case of a null-recurrent random walk,*

$$(8) \qquad \mu_j^{(0)} = q_j\{y_j + \int_{-1}^{1} Q_j(x) dF(x)\}$$

*where* $F(x)$ *is of bounded variation and* $y_j \to 0$, *then* $\mu_j^{(2n)}$ *converges for each* $j$. *The limits of* $\mu_j^{(n)}$ *exist if and only if* $F(x)$ *is continuous at* $-1$; *in any case the Cesàro limits exist.*

*Proof.* As before, it is enough in view of Theorem 2 to consider the case when $y_j = 0$ for all $j$. Using (7) we have

$$\mu_k^{(n)} = \lim_{r\to 1-} \sum_j \mu_j^{(0)} r^j P_{jk}^{(n)} = \lim_{r\to 1-} q_k \int_{-1}^{1} x^n Q_k(x) \sum_j \mu_j^{(0)} r^j Q_j(x) d\psi(x). \tag{9}$$

Putting $n = 0$ and taking note of (8) gives for each $k$

$$q_k \int_{-1}^{1} Q_k(x) dF(x) = \lim_{r\to 1-} q_k \int_{-1}^{1} Q_k(x) dF_r(x),$$

where $dF_r(x) = \sum_j \mu_j^{(0)} r^j Q_j(x) d\psi(x)$; this implies that for every polynomial $f(x)$,

$$\lim_{r\to 1-} \int_{-1}^{1} f(x) dF_r(x) = \int_{-1}^{1} f(x) dF(x).$$

Hence (9) can be rewritten as

$$\mu_k^{(n)} = q_k \int_{-1}^{1} x^n Q_k(x) dF(x)$$

from which the conclusions of the theorem are obvious.

**4. Convergence to Derman's stationary process.** In this section we assume that at time 0, there are $N_j^0$ particles in state $j$ of the Markov chain, where the random variables $N_j^0$ are independent and

$$\mu_j^{(0)} = E(N_j^0), m_j = E[N_j^0(N_j^0 - 1)].$$

The particles then evolve independently of each other according to the law of the chain, and the number in state $j$ at time $n$ is called $N_j^n$. It is easy to see from (2) that $E(N_j^n) = \mu_j^{(n)}$. It is also not hard to verify (Derman's computations in **(4)** do it) that if the $N_j^0$ are all Poisson distributed, so are the $N_j^n$. In this case the question of the existence of limiting distributions as $n \to \infty$ reduces to the existence of limits of the sequences $\mu_j^{(n)}$.

We shall investigate a more general situation:

THEOREM 6. *Suppose that for some $k$, the moments of the random variables $N_j^0$ satisfy*

$$\lim_{n\to\infty} \max_j \mu_j^{(0)} P_{jk}^{(n)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \sum_j m_j (P_{jk}^{(n)})^2 = 0. \tag{10}$$

*Suppose also that $\mu_k = \lim \mu_k^{(n)}$ exists. Then as $n \to \infty$, the distribution of $N_k^n$ approaches a Poisson distribution with mean $\mu_k$.*

*Remark.* For each $k$ such that

$$\lim_{n\to\infty} \max_j q_j P_{jk}^{(n)} = 0, \tag{11}$$

condition (10) holds provided the simpler conditions

$$|\mu_j^{(0)}| < Mq_j \quad \text{and} \quad |m_j| < Mq_j^2 \quad \text{for all} \quad j \tag{12}$$

are satisfied. For most interesting types of Markov chains, (11) holds for all $k$. However, an example of a chain with a state not satisfying (11) can be constructed along the lines of Example 4 of **(4)**.

*Proof of Theorem* 6. The proof is based on the use of generating functions. Let

$$f_k^{(n)}(x) = E[x^{N_k^n}],$$

and notice that

$$N_k^n = \sum_j N_{jk}^n,$$

where $N_{jk}{}^n$ is the number of particles which are in state $j$ at time 0 and in state $k$ at time $n$. For given $n$ and $k$ the $N_{jk}{}^n$ are independent, and $N_{jk}{}^n$ is the sum of $N_j{}^0$ independent random variables equal to 1 or 0 with probabilities $P_{jk}{}^{(n)}$ and $1 - P_{kj}{}^{(n)}$. Combining these facts gives

$$f_k^{(n)}(x) = \prod_j f_j^{(0)}(1 - P_{jk}^{(n)} + P_{jk}^{(n)}x). \tag{13}$$

Now under our assumptions,

$$f_j^{(0)}[1 - P_{jk}^{(n)}(1 - x)] = 1 - \mu_j^{(0)}P_{jk}^{(n)}(1 - x) + \tfrac{1}{2}\theta_j m_j[P_{jk}^{(n)}(1 - x)]^2 \tag{14}$$

where $0 < \theta_j < 1$. Substituting (14) in (13), taking the logarithm, and using the estimate

$$u - 1 > \log u > u - 1 - \frac{(u - 1)^2}{u}$$

for $0 < u < 1$ yields

$$\left|\log f_k^{(n)}(x) - \sum_j (x - 1)\mu_j^{(0)}P_{jk}^{(n)}\right| < \tfrac{1}{2} \sum_j \theta_j m_j[P_{jk}^{(n)}(1 - x)]^2 +$$

$$\sum_j \frac{\{(x - 1)\mu_j^{(0)}P_{jk}^{(n)} + \frac{1}{2}\theta_j m_j[P_{jk}^{(n)}(1 - x)]^2\}^2}{1 - \mu_j^{(0)}P_{jk}^{(n)}(1 - x) + \frac{1}{2}\theta_j m_j[P_{jk}^{(n)}(1 - x)]^2}. \tag{15}$$

For $n$ large, the first bounding term is arbitrarily small because of the second part of (10). The denominators of the second term are uniformly bounded away from 0 for large $n$, again by (10), and, further, it follows that the sum of the numerators in the second term approaches 0 as $n$ increases. All these estimates hold uniformly in $x$ for $0 < x < 1$. Since it was also assumed that $\mu_k{}^{(n)} \to \mu_k$, an additional estimate in (15) yields the conclusion that

$$\lim_{n\to\infty} \log f_k^{(n)}(x) = (x - 1)\mu_k$$

uniformly for $0 < x < 1$, and the theorem follows.

*Remark*. Theorems of a similar sort have been proved for spatially-homogeneous processes by Maruyama **(8)** and (more generally) by Dobrusin **(5)**. A

theorem (similar to Maruyama's) for the case of discrete time and states is found in **(9)**; a very similar result can be deduced from Theorems 2 and 6 of the present paper. Using in addition Theorem 3 allows a generalization, which overlaps with some of the results of **(5)**.*

**5. Another type of process.** In this section we suppose that a special state of $P$, say 0, has been selected and that $X_n$ particles are put in state 0 at each time $n$. As before, the $X_n$ are independent, and each particle, once introduced, independently moves subject to the law of the Markov chain. Again let $N_i^n$ denote the number of particles in state $i$ at time $n$; let $E(X_n) = a_n$. It is not hard to see that

$$E(N_i^n) = \mu_i^{(n)} = \sum_{l=0}^{n} a_l P_{0i}^{(n-l)}. \tag{16}$$

First we shall study convergence properties of the $\mu_i^{(n)}$, and then give a theorem analogous to that of the last section on the convergence of the distributions of $N_i^n$.

THEOREM 7. *If for some value of $k$, the sequence $\mu_k^{(n)}$ is Abel summable to sum $\mu_k$, then $\mu_i^{(n)}$ is summable for all $i$ to $\mu_i$, say, and there is a constant $\alpha$ such that $\mu_i = \alpha q_i$ for all $i$.*

*Proof.* Let $U_i(x) = \sum_n \mu_i^{(n)} x^n$, $A(x) = \sum_n a_n x^n$, and $P_{ij}(x) = \sum_n P_{ij}^{(n)} x^n$. From (16),

$$U_i(x) = A(x) P_{0i}(x). \tag{17}$$

From this and the hypothesis of Abel summability of $\mu_k^{(n)}$ we obtain

$$A(x) \sim \frac{\mu_k}{(1-x)P_{0k}(x)}$$

as $x \to 1-$. Therefore for any $i$,

$$U_i(x) \sim \frac{\mu_k}{(1-x)} \frac{P_{0i}(x)}{P_{0k}(x)}.$$

But it follows from Doeblin's ratio theorem **(2)** that

$$\lim_{x \to 1-} \frac{P_{0i}(x)}{P_{0k}(x)} = \frac{q_i}{q_k}.$$

Therefore $\{\mu_i^{(n)}\}$ is Abel summable to $q_i \mu_k / q_k$, which proves the theorem.

THEOREM 8. *For each $P$, there exists a monotone sequence of positive numbers $a_n$ such that $\{u_k^{(n)}\}$ converges for every $k$.*

*This result obtained by combining our theorems 3 and 6, specialized to the case of the coin-tossing process, may be compared with the example at the end of §1 of **(4)**. It can be seen that Derman's results do not contain ours, or vice versa.

*Proof*. Let us define the $a_n$ by supposing that

$$A(x) = \frac{1}{(1-x)P_{00}(x)}.$$

It is easy to verify that $a_n \downarrow 0$; in fact

$$a_n = f_{00}^{(n+1)} + f_{00}^{(n+2)} + \ldots,$$

where $f_{ij}^{(n)}$ are the first passage probabilities for $P$. In view of (17), this definition of $A(x)$ implies that $\mu_0^{(n)} = 1$ for all $n$. Now if $k \neq 0$, let ${}_0P_{0k}^{(n)}$ be the probability of a transition from state 0 to state $k$ in $n$ steps during which state 0 is not revisited **(2)**. It follows that if $k \neq 0$,

$$P_{0k}(x) = P_{00}(x)\,{}_0P_{0k}(x),$$

where ${}_0P_{0k}(x)$ is the generating function of the ${}_0P_{0k}^{(n)}$. Hence

$$u_k(x) = \frac{P_{0k}(x)}{(1-x)P_{00}(x)} = \frac{1}{1-x}\,{}_0P_{0k}(x).$$

But $\sum_n {}_0P_{0k}^{(n)} < \infty$ **(2)**, which implies that $\lim \mu_k^{(n)} = \mu_k$ exists.

Finally we study the distribution of $N_k^n$; define

$$m_l = E(X_l(X_l - 1)).$$

THEOREM 9. *Let $\{a_n\}$ be a sequence with the property specified in Theorem* 8, *and suppose that $m_n < Ma_n^2$. Then for each $k$, $N_k^n$ is asymptotically Poisson distributed; the asymptotic means are proportional to the $q_k$.*

*Proof*. Let $g_l(x)$ be the generating function of $X_l$, and again let $f_k^{(n)}(x)$ be that of $N_k^n$. Then

$$f_k^{(n)}(x) = \prod_{l=0}^{n} g_l[1 - (1-x)P_{0k}^{(n-l)}].$$

We perform upon this generating function very much the same sort of estimate which was used in the proof of Theorem 6. The result is that

$$\lim_{n\to\infty} \log f_k^{(n)}(x) = (x-1)\lim_{n\to\infty}\sum_{l=0}^{n} a_l P_{0k}^{(n-l)} = (x-1)\mu_k$$

uniformly for $x \in (0, 1)$; the theorem follows.

## References


1. S. Banach, *Théorie des Opérations Linéaires* (Warsaw, 1932).
2. K. L. Chung, *Contributions to the theory of Markov chains* I, J. Res. Nat. Bu. Stan., *50* (1953), 203–208.
3. C. Derman, *A solution to a set of fundamental equations in Markov chains*, Proc. Amer. Math. Soc., *5* (1954), 332–334.
4. ——— *Some contributions to the theory of denumerable Markov chains*, Trans. Amer. Math. Soc., *79* (1955), 541–555.
5. R. L. Dobrušin, *On Poisson's law for the distribution of particles in space*, Ukrain. Mat. Z., *8* (1956), 127–134. (In Russian; see also Math. Rev., *18*, p. 341.)
6. W. Feller, *An introduction to probability theory* (New York, 1950).
7. S. Karlin and J. McGregor, *Random walks*. Ill. J. Math., *3* (1959), 66–81.
8. G. Maruyama, *On the Poisson distribution derived from independent random walks*, Nat. Sci. Rep. Ochanomizu Univ., *6* (1955), 1–6.
9. H. Watanabe, *On the Poisson distribution*, J. Math. Soc. Japan, *8* (1956), 127–134.


*Stanford University*

# ON MAXIMAL ABELIAN SUBRINGS OF FACTORS OF TYPE $II_1$

LAJOS PUKÁNSZKY

**1. Introduction.** Although we possess a fairly complete knowledge of the abelian subrings of rings of operators in a Hilbert space which are algebraically isomorphic to the ring of all bounded operators of a finite or infinite dimensional unitary space, that is of factors of Type I, very little is known of abelian subrings of factors[1] of Type $II_1$[1]. In **(1)**, Dixmier investigated several properties of maximal abelian subrings of factors of Type II. It turned out that their structure differs essentially from that of maximal abelian subrings of factors of Type I. He showed the existence of maximal abelian subrings in approximately finite factors,[2] possessing the property that every inner*-automorphism carrying this subring into itself is necessarily implemented by a unitary operator of this subring. These maximal abelian subrings are called singular. In addition, he constructed a $II_1$ factor containing two singular abelian subrings which cannot be connected by an inner *-automorphism of this ring.[3]

The purpose of the present paper is to introduce new invariants for abelian subrings of a factor of Type $II_1$. By means of these we shall be able to show the existence of infinitely many singular maximal abelian subrings of a factor of approximately finite type which, pairwise, cannot be connected by *-automorphisms of this ring.

Indeed (cf. Lemma 1 below), we associate with every abelian subring of a $II_1$ factor an abelian ring, such that the rings corresponding to abelian subrings connected by *-automorphisms are unitarily equivalent. Since the spatial invariants of abelian rings in a Hilbert space are well known **(4)**, we obtain a useful set of invariants for the abelian subrings, with the aid of which we construct various examples.

The author is indebted to the referee for his valuable criticism; in particular Lemma 5 in its present form was suggested by him.

Received January 8, 1959.

[1]We recall that a weakly closed self-adjoint operator algebra in a Hilbert space, which contains the unit operator (that is, a ring of operators), is a factor if its center consists only of the scalar multiples of the unit operator. A factor which is not of Type I is of Type $II_1$ if all isometries contained in it are unitary transformations. For a theory of factors cf. **(2)**. When speaking simply of a ring, we always mean a ring of operators in a Hilbert space.

[2]A $II_1$ factor is of approximately finite type, if it is generated by an ascending sequence of subfactors algebraically isomorphic to the full rings of finite dimensional unitary spaces. Two $II_1$ factors of approximately finite type are algebraically isomorphic and every $II_1$ factor contains such a subfactor. For details cf. **(3)**.

[3]Cf. **(1)**. This factor is very probably not of approximately finite type.

**2. The invariants.** Let **M** be a $\mathrm{II}_1$ factor and **P** a maximal abelian subring of it. Let $\mathrm{Tr}(A)$ $(A \in \mathbf{M})$ be the canonical trace on **M**. Putting $(X, Y) = \mathrm{Tr}(XY^*)$ for $X, Y \in \mathbf{M}$, **M** becomes a pre-Hilbert space; let $\mathfrak{H}$ be the completion of **M**. If $A \in \mathbf{M}$, there exist two bounded operators $L_A$ and $R_A$ on $\mathfrak{H}$, such that for $X \in \mathbf{M}$ we have $L_A X = AX$ and $R_A X = XA$ respectively. Let us denote by $\mathbf{R}(\mathbf{P})$ the weakly closed abelian ring in $\mathfrak{H}$ generated by the sets of operators $\{L_A; A \in P\}$ and $\{R_A; A \in P\}$.

LEMMA 1. *Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be two* $\mathrm{II}_1$ *factors,* $\mathbf{P}_1 \subset \mathbf{M}_1$ *and* $\mathbf{P}_2 \subset \mathbf{M}_2$ *two maximal abelian subrings. Let* $\phi$ *be a *-isomorphic mapping from* $\mathbf{M}_1$ *onto* $\mathbf{M}_2$, *which carries* $\mathbf{P}_1$ *onto* $\mathbf{P}_2$. *Then there exists a unitary mapping of the space* $\mathfrak{H}_1$ *onto the space* $\mathfrak{H}_2$, *which carries* $\mathbf{R}(\mathbf{P}_1)$ *onto* $\mathbf{R}(\mathbf{P}_2)$.

*Proof.* For this it is enough to show the existence of a unitary mapping $U$ from $\mathfrak{H}_1$ onto $\mathfrak{H}_2$, such that

$$UL_A U^{-1} = L_{\phi(A)}$$

and

$$UR_B U^{-1} = R_{\phi(B)}$$

for $A, B \in \mathbf{M}$. By the uniqueness of the normalized traces in $\mathrm{II}_1$ factors we have $\mathrm{Tr}_1(A) = \mathrm{Tr}_2(\phi(A))$, so that putting $U(X) = \phi(X)$ $(X \in \mathbf{M}_1)$ we get an isometry between the dense linear manifolds $\mathbf{M}_1 \in \mathfrak{H}_1$ and $\mathbf{M}_2 \in \mathfrak{H}_2$, which can be extended to a unitary mapping $U$ from $\mathfrak{H}_1$ onto $\mathfrak{H}_2$. If $A \in \mathbf{M}_1, X \in \mathbf{M}_2$, then

$$UL_A U^{-1}X = UL_A\phi^{-1}(X) = U(A\phi^{-1}(X)) = \phi(A\phi^{-1}(X)) = \phi(A)X = L_{\phi(A)}X,$$

and similarly

$$UR_A U^{-1}X = R_{\phi(A)}X$$

which proves our lemma.

As a consequence of Lemma 1 we may conclude that to all spatial invariants of the ring $\mathbf{R}(\mathbf{P})$ correspond properties of **P** which are invariant under *-isomorphisms of the ring **M** which contains **P**. Now let **P** be an arbitrary abelian ring in the Hilbert space $H$. As is known **(4)**, there exists a uniquely determined sequence of mutually orthogonal projections $P_n \in \mathbf{P}(n = 1, 2, \ldots, +\infty)$ the sum of which equals unity, such that the restriction of **P** into the subspace $P_n H$ is an $n$-fold copy of a maximal abelian ring. In particular, if **P** is unitarily equivalent to $\mathbf{P}_1$, then these sequences of projections in these two rings must correspond to each other. Our next objective is to construct a sequence $\mathbf{P}_n$ $(n = 1, 2, \ldots,)$ of singular maximal abelian subrings (cf. § 1) of an approximately finite $\mathrm{II}_1$ factor so that for $\mathbf{R}(\mathbf{P}_n)$ only $P_1$ and $P_n$ differ from zero. In this case clearly $\mathbf{R}(\mathbf{P}_n)$ and $\mathbf{R}(\mathbf{P}_m)$ cannot be unitarily equivalent for $n \neq m$, and so (Lemma 1) $\mathbf{P}_n$ and $\mathbf{P}_m$ cannot be connected by a *-automorphism of **M**.

**3. Examples.** First we recall the following facts concerning the construction of factors of Type $\mathrm{II}_1$ (cf. **1**; **2**; **3**). Let $G$ be a countably infinite group, and let $L^2(G)$ be the Hilbert space of all complex-valued square summable functions on $G$. For $a \in G$ and $f(x) \in L^2(g)$ define the unitary operator $U_a$ by

$$(U_a f)(x) = f(a^{-1}x),$$

and $V_a$ by

$$(V_a f)(x) = f(xa).$$

Let $\mathbf{M}(G)$ be the operator ring generated by the set $\{U_a; a \in G\}$. Then $\mathbf{M}(G)$ is a factor of Type $\mathrm{II}_1$ if and only if every non-trivial class of conjugate elements in $G$ is infinite. If, in addition, $G$ is the union of an increasing sequence of finite subgroups, then $\mathbf{M}(G)$ is approximately finite.[4] Let $G_0 \subset G$ be an abelian subgroup and denote by $\mathbf{P}(G_0)$ the abelian ring generated by the set $\{U_a; a \in G_0\}$. Then $\mathbf{P}(G_0)$ is maximal if and only if for $a \overline{\in} G_0$ the set $\{gag^{-1}; g \in G_0\}$ is infinite. $\mathbf{P}(G_0)$ is singular if $G$ has the following property: for every element $a \overline{\in} G_0$ and arbitrary finite subset $B \subset G$ there exists an element $g_0 \in G_0$, such that $ag_0a^{-1} \overline{\in} G_0$ and from $gg_0h^{-1} = g_0$ $(g, h \in B)$ it follows that $g = h$.

Alternatively, the ring $\mathbf{M}(G)$ can be described as follows. For $f, g \in L^2(G)$ define

$$(f \times g)(x) = \sum_{y \in G} f(xy^{-1})g(y)$$

and $U_f g = f \times g$. Then $\mathbf{M}(G)$ is the collection of those operators $U_f$, for which $U_f g \in L^2(G)$ for every $g \in L^2(G)$. For such an $U_f$ its adjoint $U^*{}_f$ is $U_{\tilde{f}}$, where

$$\tilde{f}(x) = \overline{f(x^{-1})},$$

and its trace is the value of the function $f(x)$ on the unit element of $G$, or $\mathrm{Tr}(U_f) = f(e)$. Let $G_0 \subset G$ be an abelian subgroup and let us determine $\mathbf{R}(\mathbf{P}(G_0))$. If $A = U_f$ and $B = U_g$ then

$$\mathrm{Tr}(AB^*) = \mathrm{Tr}(U_f U_g^*) = \mathrm{Tr}(U_{f \times \tilde{g}}) = (f \times \tilde{g})(e) = \sum_{x \in G} f(x)\overline{g(x)}.$$

Since the set of elements in $L^2(G)$ for which $U_f$ is a bounded operator is dense in $L^2(G)$, $\hat{H}$ (cf. Lemma 1) and $L^2(G)$ can be identified so that for

$$A = U_a (a \in G) \quad (L_A)f(x) = f(a^{-1}x)$$

and $(R_A f(x) = f(xa)$ $(x \in G)$. So finally $\mathbf{R}(\mathbf{P}(G_0))$ can be identified with the ring in $L^2(G)$ generated by the set of operators

$$\{U_a, V_b; a, b \in G_0\}.$$

[4]See footnote 2.

In the following, $G$ will denote a subgroup of the affine group over a countably infinite field $K$, obtained by restricting the subgroup of dilatations to an infinite subgroup $G_0$ of the multiplicative group of non-zero elements of $K$.[5] We shall specify $K$ and this subgroup later in a way which is suitable for our purposes. Alternatively, in each case $G$ can be described as the set of all pairs $(a, \alpha)$ ($a \in G_0, \alpha \in K^+$, where we denote by $K^+$ the additive group of $K$), multiplication being defined as $(a, \alpha)\,(b, \beta) = (ab, a\beta + \alpha)$.

LEMMA 2. *Suppose that K is the union of an increasing sequence of finite subfields. Then* $\mathbf{M}(G)$ *is a* $\mathrm{II}_1$ *factor of approximately finite type.*

*Proof.* (**1**, p. 282). Observe first, that in this case $G$ is the union of an ascending sequence of finite subgroups. Therefore we have only to prove that every non-trivial class of conjugate elements in $G$ is infinite. Suppose first that $g = (a, \alpha)$, where $\alpha \neq 0$. Then $(c, 0)\,(a, \alpha)\,(c, 0)^{-1} = (a, c\alpha)$ and if $c$ runs over the elements of $G_0$ we get infinitely many elements. On the other hand, we have $(1, \gamma)\,(a, 0)\,(1, \gamma)^{-1} = (a, \gamma - a\gamma)$, therefore, if $a \neq 1$, we get again infinitely many different elements when $c$ varies over $K$, which proves our lemma.

It is of some interest to remark that Lemma 2 holds true even if $K$ is an arbitrary countable abelian field, though the proof for it is somewhat complicated.[6]

LEMMA 2′. *Let K be an arbitrary countably infinite field. Then* $\mathbf{M}(G)$ *is a* $\mathrm{II}_1$ *factor of approximately finite type.*

*Proof.* (For the following reasoning cf. (**3**, p. 793, § 5.5)). By the definition of the group $G$ the space $L^2(G)$ is the collection of all complex-valued square summable functions of the variables $a \in G_0$ and $\alpha \in K^+$. Let $X$ be the character group of $K^+$, and $\mu$ the normalized Haar measure on it, and $L^2(X)$ the Hilbert space of square integrable functions on it. For $f(a, \alpha) \in L^2(G)$ we denote by $F(a, \chi)\,(\chi \in X)$ the function on $G_0 \times X$ obtained by taking the Fourier transforms of the functions $f(a, \alpha)$ for each fixed $a \in G_0$. Since

$$\sum_{g \in G} |f(g)|^2 = \sum_{a \in G_0} \int_X |F(a, \chi)|^2 \, d\mu$$

the correspondence $f \to F$ gives a unitary mapping from $L^2(G)$ onto $L^2(G_0) \otimes L^2(X) = H$. For $a \in G_0$ let us put $\chi^a(\alpha) = \chi(a\alpha)\,(\alpha \in K^+)$; it is well known that the collection of the mappings $\chi \to \chi^a$ is a representation of $G_0$ by automorphisms of the topological group $X$ which leaves the Haar measure invariant. Moreover, since for $\alpha \neq 0$ the set $\{a\alpha; a \in G_0\}$ is infinite,

---

[5]The subgroups of dilatations and translations are the sets $\{(a, 0);\ a \in G_0\}$ and $\{(1, \gamma);\ \gamma \in K\}$. In the following we shall sometimes write simply $a$ and $\alpha$ instead of $(a, 0)$ and $(1, \gamma)$ respectively.

[6]In the course of the proof we make use of the lemma 5.2.3. of (**3**), the proof of which has not been published yet.

$G_0$ acts ergodically on $X$. For $c \in G_0$ we have $(c, 0)^{-1}(a, \alpha) = (c^{-1}a, c^{-1}\alpha)$, so that

$$(\bar{U}_c F)(a, \chi) = \sum_{\alpha \in K^+} f(c^{-1}a, c^{-1}\alpha)\alpha(\chi)$$

$$= \sum_{\alpha \in K^+} f(c^{-1}a, \alpha)(c\alpha)(\chi) = F(c^{-1}a, \chi^c)$$

so that the operator $\bar{U}_c$ in $H$ corresponding to $U_c$ is defined for $F \in H$ by $(\bar{U}_c F)(a, \chi) = F(c^{-1}a, \chi^c)$. For $\gamma \in K^+$

$$(1, \gamma)^{-1}(a, \alpha) = (a, \alpha - \gamma).$$

Introducing for a bounded measurable function $\phi(\chi)$ on $X$ the operator $L_\phi$ by $(L_\phi F)\ (a, \chi) = \phi(\chi)F(a, \chi)\ (F \in H)$, we get easily that the operator corresponding to $V_\alpha(\alpha \in K^+)$ in $H$ is $L_{\alpha(\chi)}$. Since the operators $U_a$ and $U_a$ $(a \in G_0$, $\alpha \in K^+)$ generate $\mathbf{M}(G)$, its image in $H$ is generated by the operators $\bar{U}_a$ and $L_{\alpha(\chi)}$. By virtue of the completeness of the system of characters $\{\alpha(\chi)\}$, this ring is identical with the ring generated by the operators $U_a$ and $L_\phi$, where $\phi(\chi)$ is an arbitrary bounded measurable function on $X$. But, according to a result of Murray and Neumann, operator rings represented in this form are $\mathrm{II}_1$ factors of approximately finite type (**3**, p. 787, Lemma 5.2.3).

LEMMA 3. *The ring* $\mathbf{P}(G_0)$ *is a maximal singular abelian subring of* $\mathbf{M}(G)$.

*Proof*. (cf. (**1** p. 282, ll. 22-36)).

(a) We have $(g, 0)\ (a, \alpha)\ (g, 0)^{-1} = (g, 0)\ (ag^{-1}, \alpha) = (a, g\alpha)$. Hence if $\alpha \neq 0$, varying $g$ over $G_0$ we get infinitely many different elements of the group $G$.

(b) To prove that $\mathbf{P}(G_0)$ is singular, let $B = \{(a_j; \alpha_j) = \bar{a}_j, j = 1, 2, \ldots, n\}$ be a finite subset of $G$, and $\bar{a} = (a, \alpha) \bar{\in} G_0$. For $g \in G_0$ we have

$$\bar{a}_i g \bar{a}_j^{-1} = (a_i, \alpha_i)(g, 0)(a_j, \alpha_j)^{-1} = (a_i, \alpha_i)(g, 0)(a_j^{-1}, -\alpha_j/a_j)$$

$$= (a_i, \alpha_i)(ga_j^{-1}, -g\alpha_j/a_j) = (a_i ga_j^{-1}, -g\alpha_j a_i/a_j + \alpha_i)$$

and $\bar{a}g\bar{a}^{-1} = (g, \alpha - g\alpha)$. Since $\alpha \neq 0$, this element is not in $G_0$ for $g \neq 1$. $\bar{a}_i g \bar{a}_j^{-1} = g$ implies $a_i = a_j$, and $-g\alpha_j + \alpha_i = 0$, so that if in addition for every $i, j$ and $\alpha_j \neq 0$, $g \neq \alpha_i/\alpha_j$, we have $\bar{a}_i = \bar{a}_j$.

Now we are going to find out more about the structure of the ring $\mathbf{R}(\mathbf{P}(G_0))$ (in the following denoted simply by $\mathbf{R}$), by specializing the group $G_0$ appropriately. We know (cf. above), that $\mathbf{R}$ is generated by the set of operators $\{U_g, V_h; g, h \in G_0\}$.

LEMMA 4. *Let* $n + 1$ *be the number of double cosets of* $G$ *according to the subgroup* $G_0$ $(n = 1, 2, \ldots, +\infty)$. *Then* $R$ *is the direct sum of a maximal abelian ring with an abelian ring of uniform multiplicity* $n$.

*Proof*. Let $\Gamma$ be the set of double cosets of $G$ according to $G_0$, which differ from $G_0$. For $\gamma \in \Gamma$ let us put

$$\mathfrak{M}_\gamma = \{f(x); f(x) \in L^2(G); f(x) = 0, x \overline{\in} \gamma\}$$
$$\mathfrak{M}_0 = \{f(x); f(x) = 0, x \overline{\in} G_0\}.$$

Then $L^2(G)$ is the direct sum of the mutually orthogonal subspaces $\mathfrak{M}_0$ and $\mathfrak{M}_\gamma$ ($\gamma \in \Gamma$). Evidently these subspaces are invariant with respect to the operators $U_a$, $V_b (a, b \in G_0)$ and so they all reduce the ring $\mathbf{R}$. Let $a(\gamma) = (a_\gamma, \alpha_\gamma)$ be an arbitrary element from $\gamma$. Since $\alpha_\gamma \neq 0$, for $g, g' \in G_0$ $ga(\gamma)g' = a(\gamma)$, or $(gg'a_\gamma, \alpha_\gamma g) = (a_\gamma, \alpha_\gamma)$ implies $g' = g = 1$. Therefore, if for $f \in \mathfrak{M}_\gamma$ we define the function $f'$ on $G_0 \times G_0$ by $f'(x, y) = f(xa(\gamma)y)$ $(x, y \in G_0)$, we get an isometric mapping between $\mathfrak{M}_\gamma$ and $L^2(G_0 \times G_0)$ such that to the operators $U_a$ and $V_b$ correspond translations by the same elements in $L^2(G_0 \times G_0)$ acting on $x$ and $y$ respectively. In particular, for $\gamma, \gamma' \in \Gamma$ there exists an isometric mapping between the spaces $\mathfrak{M}_\gamma$ and $\mathfrak{M}_{\gamma'}$ such that the restrictions of the operators $U_a$ and $V_b$ in these subspaces correspond to each other under this mapping. From this it follows at once that the restriction of the ring $\mathbf{R}$ in the orthogonal complement of the subspace $\mathfrak{M}_0$ is the $n$-fold copy of its restriction to any of the subspaces $\mathfrak{M}_\gamma$.

Similarly $\mathfrak{M}_0$ can be identified with the space $L^2(G_0)$.

Let $X$ be the character group of $G_0$. Then $X \times X$ is the character group of $G_0 \times G_0$, and via Fourier transforms we have an isometry between the space $L^2(G_0 \times G_0)$ and the Hilbert space of complex-valued functions $f(\phi, \psi)$ $(\phi, \psi \in X)$ square-integrable with respect to the Haar measure $\nu$ on $X \times X$. To the operators $U_a$ and $V_a$ correspond the multiplications by $a(\phi)$ and $b(\bar{\psi})$. Since these operators generate the ring of operators consisting of multiplication by any bounded measurable function on $X \times X$, we see that the restriction of $\mathbf{R}$ in $\mathfrak{M}_\gamma$ ($\gamma \in \Gamma$) is maximal. Analogous reasoning shows that the restriction of $\mathbf{R}$ in $\mathfrak{M}_0$ is maximal abelian too. In order to prove Lemma 4 it evidently suffices to show that the restriction of $\mathbf{R}$ in the space $\mathfrak{M}_0 \otimes \mathfrak{M}_\gamma$ ($\gamma$ arbitrarily chosen from $\Gamma$) contains the projection on the subspace $\mathfrak{W}_0$. By reasonings applied above this amounts to the following: Let $Z$ be the sum of the topological spaces $X$ and $X \times X$, and let $L^2(Z)$ be the Hilbert space of functions on $Z$ square-integrable with respect to a measure $\tau$, which coincides on $X$ and $X \times X$ with the Haar measures of these compact topological groups respectively. For $a, b \in G_0$ and $F(p) \in L^2(X)$ $(p \in Z)$ let us define

$$(L_{a,b}F)(p) = h_{a,b}(p)F(p)$$

where

$$h_{a,b}(p) = \begin{cases} a(\chi)\overline{b(\chi)} & \text{if} \quad p = \chi \in X \\ a(\phi)\overline{b(\psi)} & \text{if} \quad p = (\phi, \psi) \in X \times X. \end{cases}$$

Then all that we have to prove is that the ring generated by the operators $L_{a,b}$ contains the multiplication by the characteristic function of $X$. If a sequence of linear combinations of the functions $h_{a,b}(p)$ $(a, b \in G_0)$ converges

on $X \times X$ uniformly towards a continuous function, then it converges uniformly on $Z$ towards a continuous function $f(p)$ which satisfies $f(p_1) = f(p_2)$ if $p_1 = \chi \in X$ and $p_2 = (\chi, \chi) \in X \times X$. Conversely, every continuous function on $Z$ satisfying this condition can be obtained in this way. Therefore the ring generated by the operators $L_{a,b}$ contains multiplications by such functions. But since the characteristic function of the set $X \subset Z$ can be obtained as limit of a bounded sequence of them, converging almost everywhere with respect to the measure $\tau$, our lemma is proved.

In order to obtain a sequence of singular maximal abelian subrings with the desired properties in a $\mathrm{II}_1$ factor of approximately finite type, by virtue of Lemmas 2, 3, and 4 it suffices to prove the following.

LEMMA 5. *Let $n$ be a positive integer. Then there exists a field $K$ which is the union of an increasing sequence of finite subfields, a subgroup $G_0$ of the multiplicative group $K^*$ of $K$, such that if $G$ is the subgroup of the affine group over $K$ corresponding to $G_0$, the number of double cosets of $G$ according to $G_0$ equals $n + 1$.*

*Proof*. We shall perform this in two steps.

(a) Let $G_0$ be a subgroup of the multiplicative group of $K$, which has the index $n$. We show that the number of double cosets of $G$ according to $G_0$ is $n + 1$. To see this, we observe, that if $\bar{a} = (a, \alpha)$ and $\bar{b} = (b, \beta)$ are in the same double coset then $g\bar{a}g' = b$, or $(gg'a, g\alpha) = (b, \beta)$, and so $\beta = g\alpha$ $(g, g' \in G_0)$. The converse can be proved similarly.

(b)[7] According to (a) it suffices to find a $K$ and a $G_0$, such that $G_0$ has the index $n$ in $K^*$. Let $p$ be a prime number, such that $n$ is a divisor of $p-1$; let $n^s$ be the greatest power of $n$ which divides $p-1$. For an integer $k > 0$, let us denote by $F_k$ the field of order $p^k$, and by $F_k$ the multiplicative group of $F_k$. Let $k_1 < k_2 <$ be a sequence of integers relatively prime to $n$. We have

$$F_{k_1} \subset F_{k_2} \subset \dots .$$

We denote by $K$ the union of the fields

$$F_{k_i} (i = 1, 2, \dots ,).$$

We are going to show that $K$ is the direct product of a subgroup with a cyclic group of order $n^s$, from which the existence of the $G_0$ with the required properties follows at once. The number of elements of $F_k$ is

$$p^k - 1 = (p-1)(p^{k-1} + p^{k-2} + \dots + 1).$$

On the other hand,

$$p^{k_i - 1} + p^{k_i - 2} + \dots + 1 \equiv k_i \pmod{n}.$$

[7]The author is indebted to the referee for this part of the lemma which makes it possible to avoid the use of Lemma 2 and hence that of 5.2.3 in **(3)**.

So $n^s$ is the highest power of $n$ dividing $p^{k_i}-1$. Since $F^{\times}{}_k$ is a cyclic group, it contains a cyclic group $H$ of order $n^s$, which is a direct factor, and which does not change with $i$. So $H$ is a direct factor in $K^{\times}$, which proves our lemma.

We sum up the preceding lemmas in the following theorem:

THEOREM. *If $M$ is an approximately finite factor of Type* $\mathrm{II}_1$ *then it contains an infinite sequence of singular maximal abelian subrings which cannot be pairwise connected by *-automorphisms* of $M$.

REFERENCES

1. J. Dixmier, *Sous-anneaux abéliens maximaux dans les facteurs de type fini*, Ann. Math., *59* (1957), 279–286.
2. F. J. Murray and J. von Neumann, *On rings of operators*, Ann. Math., *37* (1936), 116–229.
3. ——— *On rings of operators IV*, Ann. Math., *44* (1943), 716–808.
4. I. E. Segal, *Decompositions of operator algebras II*, Mem. Amer. Math. Society, no. 9 (1951).

*RIAS, Baltimore*

# LATTICE OCTAHEDRA

L. J. MORDELL

Let $A_1, A_2, \ldots, A_n$ be $n$ linearly independent points in $n$-dimensional Euclidean space of a lattice $\Lambda$. The points $\pm A_1, \pm A_2, \ldots, \pm A_n$ define a closed $n$-dimensional octahedron (or "cross polytope") $K$ with centre at the origin $O$. Our problem is to find a basis for the lattices $\Lambda$ which have no points in $K$ except $\pm A_1, \pm A_2, \ldots, \pm A_n$.

Let the position of a point $P$ in space be defined vectorially by

$$P = p_1A_1 + p_2A_2 + \ldots + p_nA_n, \tag{1}$$

where the $p$ are real numbers. We have the following results.

When $n = 2$, it is well known that a basis is

$$(A_1, A_2). \tag{2}$$

When $n = 3$, Minkowski **(1)** proved that there are two types of lattices, with respective bases

$$(A_1, A_2, A_3),\ (A_1, A_2, \tfrac{1}{2}(A_1 + A_2 + A_3)). \tag{3}$$

When $n = 4$, there are six essentially different bases typified by $A_1, A_2, A_3$ and one of

$$\begin{array}{ll} A_4, \tfrac{1}{2}(A_2 + A_3 + A_4), & \tfrac{1}{2}(A_1 + A_2 + A_3 + A_4), \\ \tfrac{1}{3}(\pm A_1 \pm A_2 \pm A_3 \pm A_4), & \tfrac{1}{4}(\pm 2A_1 \pm A_2 \pm A_3 \pm A_4), \\ \tfrac{1}{5}(\pm 2A_1 \pm 2A_2 \pm A_3 \pm A_4). & \end{array} \tag{4}$$

In all expressions of this kind, the signs are independent of each other and of any other signs. This result is a restatement of a result by Brunngraber **(2)** and a proof is given by Wolff **(3)**.

The proofs for $n = 3, 4$ depend upon Minkowski's method of adaption of lattices, and that for $n = 4$ is very complicated. I notice another method of considering the question which gives the result more directly, more simply, and with less troublesome numerical detail.

The simplest required lattice is that with basis $(A_1, A_2, \ldots, A_n)$. This will not be a basis of the other lattices $\Lambda$. Hence there will be points $A$ of $\Lambda$ given by

$$pA = a_1A_1 + a_2A_2 + \ldots + a_nA_n, \tag{5}$$

where $a_1, a_2, \ldots, a_n$ and $p > 1$ are integers, and

$$(a_1, a_2, \ldots, a_n, p) = 1. \tag{6}$$

Received January 28, 1959.

For brevity, we shall denote such a point $A$ by

$$A = \{a_1, a_2, \ldots, a_n\}/p.$$

There is no loss of generality in supposing that

$$|a_1| \leqslant \tfrac{1}{2}p, \; |a_2| \leqslant \tfrac{1}{2}p, \ldots, |a_n| \leqslant \tfrac{1}{2}p. \tag{7}$$

We may also suppose that no $a \equiv 0 \pmod{p}$. For if $a_1 \equiv 0 \pmod{p}$, we have an $n - 1$ dimensional problem which may be considered as solved in dealing with the $n$-dimensional problem.

By the conditions of the problem, the point $A$ is such that for any integer $x$ prime to $p$, and all integers $x_1, x_2, \ldots, x_n$

$$xA - x_1A_1 - x_2A_2 - \ldots - x_nA_n$$

is not in $K$; and there is no loss of generality in supposing that $|x| < p$. We shall call such points $A$ admissible. Then $A$ will be admissible if and only if

$$\left|\frac{a_1x}{p} - x_1\right| + \ldots + \left|\frac{a_nx}{p} - x_n\right| \geqslant 1, \tag{8}$$

since the point $P$ in (1) lies in $K$ if

$$|p_1| + |p_2| + \ldots + |p_n| < 1. \tag{9}$$

Now by Minkowski's theorem on convex bodies, the convex $n + 1$ dimensional body

$$|X_1| + |X_2| + \ldots + |X_n| < 1, \; |X| < p$$

of volume $2^{n+1}p/n!$ contains at least two points of the lattice given by

$$X_1 = \frac{a_1x}{p} - x_1, \ldots, X_n = \frac{a_nx}{p} - x_n, X = x,$$

of determinant one when $p > n!$ We may suppose that $X \neq 0$ since then $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. Hence, as is well known, admissible points $A$ can arise only when $p \leqslant n!$

In this paper, we shall be concerned only with the cases $n = 2, 3, 4$. We shall see that admissible points $A$ arise only when $n = 3$, $p = 2$, and $n = 4$, $p = 2, 3, 4, 5$.

Suppose first that $n = 2$. We need only consider $p = 2$, and then $|a_1| \leqslant 1$, $|a_2| \leqslant 1$. Clearly the point $A = \frac{1}{2}\{a_1, a_2\}$ lies in $K$ and so cannot be a point of $\Lambda$. Hence $(A_1, A_2)$ is a basis of $\Lambda$.

Suppose next that $n = 3$. We have now to consider $p = 2, 3, 4, 5, 6$.

*If* $p = 2$, $|a_1| \leqslant 1$, $|a_2| \leqslant 1$, $|a_3| \leqslant 1$, and then $A = \frac{1}{2}\{a_1, a_2, a_3\}$. This will be a point of $K$ unless $|a_1| = |a_2| = |a_3| = 1$, and so $A = \frac{1}{2}\{\pm 1, \pm 1, \pm 1\}$. This point is admissible since $xA \equiv A \pmod{\Lambda}$ when $x = \pm 1$. Hence we clearly have a lattice $\Lambda$ typified by the basis $(A = \frac{1}{2}\{1, 1, 1\}, A_1, A_2)$, since $A_3 = 2A - A_1 - A_2$.

*If* $p = 3$, $|a_1| \leqslant 1$, $|a_2| \leqslant 1$, $|a_3| \leqslant 1$, then $A = \frac{1}{3}\{\pm 1, \pm 1, \pm 1\}$ and lies in $K$ and is not admissible.

*If* $p = 4$, $|a_1| \leqslant 2$, $|a_2| \leqslant 2$, $|a_3| \leqslant 2$. We may suppose that one at least of the $a$'s is not even, say $|a_1| = 1$. Since $A$ does not lie in $K$, the only possibility for $A$ is $A = \frac{1}{4}\{\pm 1, \pm 2, \pm 2\}$. Then $2A \equiv \frac{1}{2}A_1 \pmod{\Lambda}$ and so $A$ is not admissible.

*If* $p = 5$, $|a_1| \leqslant 2$, $|a_2| \leqslant 2$, $|a_3| \leqslant 2$ and so since $A$ is not in $K$, we must have $A = \frac{1}{5}\{\pm 2, \pm 2, \pm 2\}$. Then $2A \equiv \frac{1}{5}\{\pm 1, \pm 1, \pm 1\} \pmod{\Lambda}$, and so $A$ is not admissible since $\frac{1}{5}\{\pm 1, \pm 1, \pm 1\}$ lies in $K$.

*If* $p = 6$, $|a_1| \leqslant 3$, $|a_2| \leqslant 3$, $|a_3| \leqslant 3$. Since we require $|a_1| + |a_2| + |a_3| > 6$, we have only the three cases typified by

$$(a_1, a_2, a_3,) = (\pm 1, \pm 3, \pm 3), \quad (\pm 2, \pm 2, \pm 3), \quad (\pm 2, \pm 3, \pm 3), \quad (\pm 3, \pm 3, \pm 3).$$

In all these, $2A$ is congruent mod $\Lambda$ to a point of $K$ and so $A$ is not admissible.

Suppose finally that $n = 4$ and so now $p \leqslant 24$. We shall show that there exist admissible points if and only if $p \leqslant 5$. We first give some results of a general character which will simplify the arithmetic. We note

(I) $A$ is not admissible if $p$ contains a factor $f$ such that every $A$ with denominator $f$ is not admissible. This is obvious from

$$pA/f = \{a_1, a_2, a_3, a_4,\}/f.$$

We note next

(II) $A$ is not admissible if for $d$, the greatest common divisor of $p$ and of any of the $a$'s, $d > 2$.

For suppose that $(a_1, p) = d$. Then $pA/d \equiv \{0, a_2, a_3, a_4\}/d \pmod{\Lambda}$, and from the case $n = 3$, this cannot be admissible unless $d = 2$ and $a_2, a_3, a_4$ are all odd. Hence, whenever $A$ is admissible, we may suppose that one of the $a$, say $a_1$ is odd and prime to $p$. On considering $xA$ where $xa_1 \equiv \pm 1 \pmod{p}$, we may then take

(III) $a_1 \equiv \pm 1 \pmod{p}$.

We shall presently consider the admissible points with $|a_2| = 1, 2, 3$, but first we consider the smaller values of $p$.

*When* $p = 2, 3$, it is clear that the only admissible points $A$ are

$$A = \{\pm 1, \pm 1, \pm 1, \pm 1\}/p.$$

Note $Ax \equiv \{\pm 1, \pm 1, \pm 1, \pm 1\}/p \pmod{\Lambda}$ for $x = \pm 1$.

*When* $p = 4$, $|a_1| \leqslant 2$, $|a_2| \leqslant 2$, $|a_3| \leqslant 2$, $|a_4| \leqslant 2$. Since A is admissible, $\sum|a| > 5$, and since all the $|a|$ cannot be less than 2, we can take say $|a_1| = 2$. Then from (II), $a_2, a_3, a_4$ are odd giving the admissible point

$$A = \tfrac{1}{4}\{\pm 2, \pm 1, \pm 1, \pm 1\}.$$

We note $2A \equiv \frac{1}{2}\{0, 1, 1, 1\} \pmod{\Lambda}$.

*When* $p = 5$, $|a_1| < 2$, etc. We can take $|a_1| = 1$, and since $\sum|a| > 6$, we may take, say, $|a_2| = 2$, and then, say, $|a_3| = 2$. We can reject $|a_4| = 2$ since for $A = \frac{1}{5}\{\pm 1, \pm 2, \pm 2, \pm 2\}$, $2A$ is not admissible. When $|a_4| = 1$, we have the admissible point $A$ typified by

$$A = \tfrac{1}{5}\{\pm 2, \pm 2, \pm 1, \pm 1\}.$$

We note $2A \equiv \frac{1}{5}\{\pm 1, \pm 1, \pm 2, \pm 2\} \pmod{\Lambda}$.

*When* $p = 6$, by means of (II), we can exclude the cases when any $a$ is divisible by 3, and also when any $a$ is divisible by 2, since then the only possible forms for $A$ are given by $A = \frac{1}{6}\{\pm 2, \pm 1, \pm 1, \pm 1\}$, and these are obviously not admissible. Hence also from (I),

$$p = 12, 18, 24 \text{ are not admissible.}$$

*When* $p = 7$, we have $|a_1| = 1$ and then, say, $|a_2| = 3$. Hence $|a_3| = 2$ or 3. We reject $|a_3| = 3$ since then $2A \equiv \frac{1}{7}\{\pm 2, \pm 1, \pm 1, 2a_4\} \pmod{\Lambda}$ and is inadmissible. Then $|a_4| = 2$ or 3 and we can reject $|a_4| = 3$ leaving $A = \frac{1}{7}\{\pm 1, \pm 3, \pm 2, \pm 2\}$; and $3A \equiv \frac{1}{7}\{\pm 3, \pm 2, \pm 1, \pm 1\} \pmod{\Lambda}$ and is not admissible. Hence also

$$p = 7, 14, 21 \text{ are not admissible.}$$

*When* $p = 8$, suppose first that all the $a$ are odd. Since $|a| = 1$ or 3, at least two of the $|a|$ are equal, and on considering $3A$, if need be, we can take $|a_1| = 1$, $|a_2| = 1$. Then $A = \frac{1}{8}\{\pm 1, \pm 1, a_3, a_4\}$ is obviously inadmissible: Suppose next that some of the $a$ are even. Then by (II), we need only consider the case when $|a_1| = 2$, and $|a_2|, |a_3|, |a_4|$, are odd. Since at least two of these are equal, we may on considering $3A$ if need be, take $|a_2| = 1, |a_3| = 1$ and then $A$ is inadmissible: Hence also

$$p = 16, 24, \text{ are not admissible.}$$

*When* $p = 9$, on considering $3A$, we see that each $a$ satisfies $a \equiv \pm 1 \pmod 3$, that is, $|a| = 1$, 2, or 4. Since at least two of the $|a|$ are equal, we can on considering $2A$ or $4A$, if need be, take $|a_1| = 1$, $|a_2| = 1$. Hence $A = \frac{1}{9}\{\pm 1, \pm 1, \pm 4, \pm 4\}$, and $2A$ is not admissible. Hence also

$$p = 18, \text{ is not admissible.}$$

*When* $p = 10$, we have $|a_1| = 1$, and since $|a_2| + |a_3| + |a_4| > 10$, we must have, say, $|a_4| = 4$ or 5. By (II), we can reject $|a_4| = 5$, and when $|a_4| = 4$, $a_3$ and $a_4$ must be odd and so $|a_3| < 3$, $|a_4| < 3$. The only possibility is $A = \frac{1}{10}\{\pm 1, \pm 4, \pm 3, \pm 3\}$, but then $3A$ is not admissible. Hence also

$$p = 20 \text{ is not admissible.}$$

We have now dealt with all the even values of $p < 24$, except $p = 22$ which will be dealt with when $p = 11$ is considered, and which is not admissible. We must now consider the remaining odd values of $p > 9$. We shall show that

no admissible points $A$ arise when $p > 5$ and $|a_1| = 1$, $|a_2| = 1, 2$, or 3. This will then hold also for any two $a$, say $a_r$, $a_s$ if $(a_r, p) = 1$ and $a_s \equiv \pm a_r$, $\pm 2a_r$, $\pm 3a_r \pmod p$.

(IV) Suppose $|a_1| = 1$, $|a_2| = 1$. Since $|a_3| + |a_4| > p - 1$, we must have $|a_3| = \frac{1}{2}(p-1)$, $|a_4| = \frac{1}{2}(p-1)$. Then $2A \equiv \{\pm 2, \pm 2, \pm 1, \pm 1\}/p \pmod \Lambda$ and $2A$ is not admissible if $p > 7$.

(V). Suppose $|a_1| = 1$, $|a_2| = 2$. Then $|a_3| + |a_4| > p - 2$ and so, say, $|a_3| = \frac{1}{2}(p-1)$. Then $|a_4| = \frac{1}{2}(p-1)$ or $\frac{1}{2}(p-3)$. The first value can be rejected by (IV) since

$$\left(\frac{p-1}{2}, p\right) = 1.$$

For the second, $2A = \{\pm 2, \pm 4, \pm 1, \pm 3\}/p$ and is not admissible if $p > 11$. We have seen that no admissible points arise when $p = 7$ or 9.

(VI). Suppose finally $|a_1| = 1$, $|a_2| = 3$. Since $|a_3| + |a_4| > p - 3$, we have $|a_3| = \frac{1}{2}(p-1)$ or $\frac{1}{2}(p-3)$. Since $a_1 \equiv \pm 2a_3$, we need only consider $|a_3| = \frac{1}{2}(p-3)$ and then $|a_4| = \frac{1}{2}(p-3)$. This can be rejected by (IV) when $(p, 3) = 1$, and by (II) when $(p, 3) = 3$.

We now consider the odd values of $p > 11$. We know from (IV), (V), and (VI), that we need consider only the cases when $|a_1| = 1$, and the other $a$ satisfy $|a| > 4$; and of course all $a$ satisfy $|a| < \frac{1}{2}p$. We can reject all $a = \pm \frac{1}{2}(p-1)$ or $a = \pm \frac{1}{3}(p-1)$.

$p = 11$. Here $|a_2| = 4$ or 5, and both can be rejected. Hence $A$ is not admissible.

$p = 13$. Here $|a_2| = 4, 5$, or 6 and $|a_2| = 4, 6$ can be rejected, and so $|a_2| = 5$. Since $|a_3| = 4, 5$, or 6, we can reject 4, 6 and then $|a_2| = |a_3|$. Hence $A$ is not admissible.

$p = 15$. Here $|a_2| = 4, 5, 6, 7$ and we can reject 5, 6, and also 7 from (II). Hence $|a_2| = 4$ and this is also the only possibility for $|a_3|$. Hence $A$ is not admissible.

$p = 17$. Here $|a_2| = 4, 5, 6, 7, 8$ and we can reject 6, 8. Since $|a_2|, |a_3|, |a_4|$ are distinct by (IV), they must be 4, 5, 7 in some order, and then $|a_1| + |a_2| + |a_3| + |a_4| = 17$, so that $A$ is not admissible.

$p = 19$. Here $|a_2| = 4, 5, 6, 7, 8, 9$.

We can reject 6 and 9. Hence $|a_2|, |a_3|, |a_4|$ are three out of 4, 5, 7, 8 and since $|a_2| + |a_3| + |a_4| > 19$, we can suppose that $A = \{\pm 1 \pm 7, \pm 8, a_4\}/19$ where $a_4 = \pm 4$ or $\pm 5$. But now $3A \equiv \{\pm 3, \pm 2, \pm 5, \pm 3a_4\}/19 \pmod \Lambda$ and is not admissible since $3a_4 \equiv \pm 7$ or $\pm 4 \{\bmod 19\}$.*

$p = 23$. Here $|a_2| = 4, 5, 6, 7, 8, 9, 10, 11$.

We can reject 8, 11. The cases $|a_2| = 6, 9, 10$ are included under $|a_2| = 4, 5, 7$ respectively on considering $4A$, $5A$, $7A$, respectively.

*I am indebted to the referee for these proofs for $n = 17,19$, which are rather shorter than those I had given.

When $|a_2| = 4$, $|a_3| + |a_4| > 19$ and so $|a_3| = 10$. Then $A = \{\pm 1, \pm 4, \pm 10, a_4\}/23$, and $5A \equiv \{\pm 5, \pm 3, \pm 4, 5a_4\}/23 \pmod{\Lambda}$ is not admissible.

When $|a_2| = 5$, $|a_3| + |a_4| > 18$ or, say, $|a_3| = 9, 10$. We can reject 10 since $|a_3| = 2|a_2|$. Hence $A = \{\pm 1, \pm 5, \pm 9, a_4\}$ and now $3A \equiv \{\pm 3, \pm 8, \pm 4, 3a_4\}$ is not admissible.

When $|a_2| = 7$, $|a_3| + |a_4| > 16$ and so $|a_3| = 9, 10$ and so $A = \{\pm 1, \pm 7, \pm 9 \text{ or } \pm 10, a_4\}/23$. Now $7A \equiv \{\pm 7, 3, \pm 6, \text{or } \pm 1, 7a_4\}/23 \pmod{\Lambda}$ and is clearly not admissible.

We can now find the possible bases for $\Lambda$. We may suppose that not all of the bases of the three-dimensional sublattices are of the type $(A_1, A_2, \frac{1}{2}(A_1 + A_2 + A_3))$. For if $(A_1, A_2, \frac{1}{2}(A_1 + A_2 + A_4))$ were also allowable, then $\frac{1}{2}(A_3 - A_4)$ would be a point of $\Lambda$. Hence we may suppose that three of the $A$'s, say, $A_1, A_2, A_3$ form a basis for the three-dimensional sublattice. Then the fourth basis element $A$ must be such that $A_4 = bA + b_1A_1 + b_2A_2 + b_3A_3$ where the $b$ are integers. Clearly we can typify $A$ by one of $A_4, \frac{1}{2}(A_2 - A_3 - A_4)$ and $\frac{1}{2}(1, 1, 1, 1\}, \frac{1}{3}(\pm 1, \pm 1, \pm 1, \pm 1\}, \frac{1}{4}(\pm 2, \pm 1, \pm 1, \pm 1\}, \frac{1}{3}\{\pm 2, \pm 2, \pm 1, \pm 1\}$.

This completes the proof for $n = 4$. We note that we have shown that when $n = 4$, integers $x, x_1, x_2, \ldots, x_n$ not all zero exist for which

$$\left|\frac{a_1x}{p} - x_1\right| + \ldots + \left|\frac{a_nx}{p} - x_n\right| < 1, \; |x| < p$$

not only when $p > 4!$ but also when $p > 5$. It is an interesting problem to find the exact result for $n > 4$. Approximate results for large $n$ have been given by Blichfeldt **(4)**.

## References


1. H. Minkowski, *Gesammelte Abhandlungen*, Bd. II.
2. E. Brunngraber, *Ueber Punktgitter* (Dissertation, Wien, 1944).
3. K. M. Wolff, Monatsh. Math., *58* (1954), 38–56.
4. H. F. Blichfeldt, Monatsh. Math., *42–43* (1935–6), 410–414.



*Mount Allison University Sackville, New Brunswick, Canada*
*St. Johns College, Cambridge, England*

# A NOTE ON CONTINUED FRACTIONS

A. OPPENHEIM

**1. Introduction.** Any real number $y$ leads to a continued fraction of the type

$$(1) \qquad y \sim b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots,$$

where $a_i$, $b_i$ are integers which satisfy the inequalities

$$(2) \qquad 1 \leqslant a_i \leqslant b_i \qquad (i = 1, 2, \ldots,),$$

by means of the algorithm

$$(3) \qquad \begin{aligned} y = y_0 &= [y_0] + \frac{a_1}{y_1} = b_0 + \frac{a_1}{y_1}, y_1 > a_1, \\ y_1 &= [y_1] + \frac{a_2}{y_2} = b_1 + \frac{a_2}{y_2}, y_2 > a_2, \end{aligned}$$

the $a$'s being assigned positive integers. The process terminates for rational $y$; the last denominator $b_k$ satisfying $b_k \geqslant a_k + 1$. For irrational $y$, the process does not terminate. For a preassigned set of numerators $a_i \geqslant 1$, this C.F. development of $y$ is unique; its value being $y$.

Bankier and Leighton **(1)** call such fractions (1), which satisfy (2), proper continued fractions. Among other questions, they studied the problem of expanding quadratic surds in periodic continued fractions. They state that "it is well-known that not only does every periodic regular continued fraction represent a quadratic irrational, but the regular continued fraction expansion of a quadratic irrational is periodic. Such a result would not be expected to hold in general for proper continued fraction representations of quadratic irrationals" **(1**, p. 662**)**.

In point of fact, as I prove in this note, *every* quadratic irrational admits of infinitely many periodic proper continued fraction representations. Indeed, only one term is needed in the periodic part and at most three terms in the non-periodic part:

$$(4) \qquad b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \left(\frac{a_3}{b_3}\right)_\infty.$$

Moreover, in infinitely many representations $a_3 = b_3 = 2c$ or, again, with $a_3 = c$, $b_3 = 2c$. For the class of quadratic irrationals whose regular continued fraction expansion has a period with an *odd* number of terms, it is possible to have (in infinitely many ways) $a_3 = 1$, $b_3 = 2c$.

Received March 17, 1959.

It may be noted that Bankier and Leighton obtained periodic proper continued fraction expansions in infinitely many ways for the class of quadratic irrationals whose regular continued fraction expansion (i) is purely periodic and (ii) has an odd number of terms in the period. My results are stated explicitly in the following three theorems.

THEOREM 1. *Any real quadratic irrational can be expressed as a proper periodic continued fraction of the form* (4), *in which the period consists of one term only and the non-periodic part (which may be empty) contains at most three terms.*

*The expansion is possible, in infinitely many ways, with $b_3 = 2c$, an even integer.*

THEOREM 2. *For a given quadratic irrational, there are infinitely many of the expansions in Theorem* 1 *satisfying*

$$a_3 = b_3 = 2c.$$

*This is also true with $a_3 = c$, $b_3 = 2c$.*

THEOREM 3. *Let $\theta$ be any quadratic irrational and write*

$$\theta = b_0 + \zeta_0, \qquad b_0 = [\theta], \qquad 0 < \zeta_0 < 1,$$

*where*

$$\zeta_0 = \frac{P_0 \pm \sqrt{N_0}}{R_0}.$$

*Let $s_0$ be the least positive integer satisfying*

$$R_0 | s_0(P_0^2 - N_0).$$

*Then, if the regular continued fraction for $s_0\sqrt{N_0}$ has a period with an odd number of elements, infinitely many of the expansions* (4) *for $\theta$, have $a_3 = 1$.*

For the proofs we require five lemmas and these are stated and proved in §§ 3, 4.

**2. A conjecture.** Some time ago I made a conjecture (which I cannot prove) about these representations, when the integers $a_i$ are assigned in a special way. Let $y > 1$ be an irrational number. Assign $a_1 = b_0$. Determine $b_1$ and assign $a_2 = b_1$. Determine $b_2$ and assign $a_3 = b_2$, and so on. In this way, we determine a *unique* expansion

$$y = b_0 + \frac{b_0}{b_1 +}\,\frac{b_1}{b_2 +}\,\frac{b_2}{b_3 + \dots}, \tag{5}$$

in which the integers $b_i$ satisfy the inequalities

$$1 < b_0 < b_1 < b_2 < \dots. \tag{6}$$

Plainly, if $b_i = b$ from some point on, $y$ must be a quadratic surd.

CONJECTURE. *If $y > 1$ is a quadratic irrational, the expansion* (5) *is ultimately periodic, that is, from some point on, the $b_i$ have a fixed value.*

*Examples:*

$$\frac{2\sqrt{2}+1}{3} = 1 + \frac{1}{3} + \frac{3}{4} + \left(\frac{4}{4}\right)_\infty,$$

$$\frac{b}{a}(a^2+1)^{\frac{1}{2}} = b + \frac{b}{2a^2} + \frac{2a^2}{4a} + \left(\frac{4a^2}{4a^2}\right)_\infty, \qquad (b = 1, 2, \ldots, 2a^2),$$

$$\frac{b}{a}(a^2+2a)^{\frac{1}{2}} = b + \frac{b}{a} + \frac{a}{2a} + \left(\frac{2a}{2a}\right)_\infty, \qquad (b = 1, 2, \ldots, a),$$

$$\frac{b}{3a}(9a^2+3)^{\frac{1}{2}} = b + \frac{b}{6a^2} + \frac{6a^2}{12a^2} + \left(\frac{12a^2}{12a^2}\right)_\infty, \qquad (b = 1, 2, \ldots, 6a^2).$$

**3.** In what follows, $N$ is a positive non-square integer,

$$c = [N^{\frac{1}{2}}], \qquad N = c^2 + a, \qquad 1 < a < 2c. \tag{7}$$

We express each of $N^{\frac{1}{2}}$, $-N^{\frac{1}{2}}$ as proper continued fractions with a single term in the periodic part and then apply the results to the general quadratic irrational.

LEMMA 1. *Let*

$$\xi = \frac{a}{2c} + \frac{a}{2c} + \frac{a}{2c} + \ldots.$$

*Then*

$$\xi = -c + N^{\frac{1}{2}}.$$

*Proof.* Since $\xi > 0$, $\xi(2c + \xi) = a$, we have

$$\xi = -c + (c^2 + a)^{\frac{1}{2}}.$$

LEMMA 2. *Let*

$$\eta = \frac{2c+1-a}{2c+1} + \frac{a}{2c} + \frac{a}{2c} + \ldots.$$

*Then*

$$\eta = c + 1 - N^{\frac{1}{2}}.$$

*Proof.* Note that

$$\eta = \frac{2c+1-a}{2c+1+\xi} = \frac{(c+1)^2 - N}{c+1+N^{\frac{1}{2}}} = c + 1 - N^{\frac{1}{2}}.$$

**4.** We next consider quadratic irrationals of the following three types:

$$\text{I.}\quad \zeta = \frac{P + N^{\frac{1}{2}}}{R};0 < \zeta < 1, R \geqslant 1, -N^{\frac{1}{2}} < P < N^{\frac{1}{2}}, R|(N - P^2),$$

$$\text{II.}\quad \zeta = \frac{P + N^{\frac{1}{2}}}{R};0 < \zeta < 1, R \geqslant 1, P > N^{\frac{1}{2}}, R|(P^2 - N),$$

$$\text{III.}\quad \zeta = \frac{P - N^{\frac{1}{2}}}{R};0 < \zeta < 1, R \geqslant 1, R|(P^2 - N).$$

LEMMA 3. *For surds of type I, define integers $a_1$, $b_1$, by the conditions*

$$a_1R = N - P^2, \qquad b_1 = [N^{\frac{1}{2}} - P] = c - P.$$

*Then*

$$\zeta = \frac{a_1}{b_1+}\,\frac{a}{2c+}\,\frac{a}{2c+}\dots = \frac{a_1}{b_1 + \xi}.$$

*Proof.* Note that

$$\frac{P + N^{\frac{1}{2}}}{R} = \frac{a_1}{-P + N^{\frac{1}{2}}},$$

where

$$0 < \frac{a_1}{N^{\frac{1}{2}} - P} < 1, N^{\frac{1}{2}} - P > 0.$$

Hence $1 \leqslant a_1 < N^{\frac{1}{2}} - P$ or $a_1 \leqslant c - P = b_1$. Thus

$$\frac{a_1}{b_1 + \xi} = \frac{a_1}{b_1 - c + N^{\frac{1}{2}}} = \frac{a_1}{N^{\frac{1}{2}} - P} = \zeta,$$

and the result follows from Lemma 1.

LEMMA 4. *For surds of type* II *define integers $a_2$, $b_2$ by the conditions*

$$a_2R = P^2 - N, \qquad b_2 = [P - N^{\frac{1}{2}}] = P - c - 1.$$

*Then*

$$\zeta = \frac{a_2}{b_2 + \eta} = \frac{a_2}{b_2+}\,\frac{2c + 1 - a}{2c + 1}\,\frac{a}{+2c}\,\frac{a}{+2c}+\dots.$$

*Proof.* Observe that

$$\zeta = \frac{P + N^{\frac{1}{2}}}{R} = \frac{a_2}{P - N^{\frac{1}{2}}},$$

where

$$0 < \frac{a_2}{P - N^{\frac{1}{2}}} < 1, P - N^{\frac{1}{2}} > 0.$$

Hence

$$1 \leqslant a_2 < P - N^{\frac{1}{2}}, \qquad 1 \leqslant a_2 \leqslant P - c - 1 = b_2.$$

Thus our continued fraction for $\zeta$ is proper and its value is clearly

$$\frac{a_2}{b_2+\eta}=\frac{a_2}{b_2+c+1-N^{\frac{1}{2}}}=\frac{a_2}{P-N^{\frac{1}{2}}}.$$

LEMMA 5. *For surds of type* III *define an integer* $a_2$ *by the condition* $a_2R = P^2 - N$. *Then*

$$\zeta=\frac{a_2}{(P+c)+\xi}=\frac{a_2}{P+c+}\frac{a}{2c+}\frac{a}{2c+}\cdots$$

*Proof.* Since

$$\zeta=\frac{a_2}{P+N^{\frac{1}{2}}}=\frac{a_2}{(P+c)+\xi},$$

where $0<\zeta<1$, $1\leqslant a_2<P+N^{\frac{1}{2}}$, $1\leqslant a_2<P+c$, the result follows from Lemma 1.

**5. Proofs of theorems 1, 2, and 3.** Let the quadratic irrational $\theta$, say, be expressed in the form $\theta = b_0+\zeta_0$, where $b_0=[\theta]$, $0<\zeta_0<1$ and

$$\zeta_0=\frac{P_0\pm N_0^{\frac{1}{2}}}{R_0}.$$

Since we can also express $\zeta_0$ in the form $(P\pm N^{\frac{1}{2}})/R$, where

$$R=sR_0,\qquad P=sP_0,\qquad N=s^2N_0\qquad(s\geqslant 1)$$

it belongs to one of the types I, II, or III, provided that the integer $s$ is chosen so that $R|(P^2-N)$. It is sufficient, then, if $R_0|s(P_0^2-N)$, and plainly $s$ can be so chosen in infinitely many ways ($s=tR_0/g$, where $g$ is the greatest common divisor of $R_0$ and $P_0^2-N_0$ and $t$ is any integer $\geqslant 1$). Thus Theorem 1 follows immediately from Lemmas 3, 4, and 5.

Observe that $s_0=R_0/(R_0,P_0^2-N_0)$ is the least positive integer such that $R_0|s_0(P_0^2-N_0)$. Then we may write

$$\zeta_0=\frac{P_0\pm N_0^{\frac{1}{2}}}{R_0}=\frac{P\pm N^{\frac{1}{2}}}{R}$$

where $N=t^2s_0^2N_0$ $(t\geqslant 1)$. Recall that

$$c^2+a=N=t^2s_0^2N_0$$

where

$$c=[N^{\frac{1}{2}}],\qquad 1\leqslant a\leqslant 2c,$$

by (7). To obtain $a=2c$, it is enough to solve the Pellian equation

$$(c+1)^2-t^2s_0^2N_0=1$$

for $c$ and $t$. Since $s_0^2N_0$ is not a square, there are infinitely many such pairs. Similarly, for $a=c$ we require the solutions of the Pellian equation

$$(2c+1)^2-4t^2s_0^2N_0=1,$$

which also gives infinitely many pairs. This proves Theorem 2.

For Theorem 3 we require $a = 1$ and then it is enough to solve the Pellian equation

$$c^2 - s_0^2 N_0 t^2 = -1.$$

Now, it is well known that if the regular continued fraction for $s_0 N_0^{\frac{1}{2}}$ has a period with an *odd* number of elements, then this has infinitely many solutions in $c$ and $t$. This proves Theorem 3.

**6.** Examples for the well-known quadratic irrational $\frac{1}{2}(1 + \sqrt{5})$ are listed.

(i) $\frac{1}{2}(1 + \sqrt{5}) = 1 + \dfrac{2t}{c + t +} \left(\dfrac{2c}{2c}\right)_\infty,$

where $c$ and $t$ satisfy $(c + 1)^2 - 5t^2 = 1$, so that

$$(c + 1) + t\sqrt{5} = (2 + \sqrt{5})^{2n}, \quad (n = 1, 2, \ldots,),$$

for example,

$$(c, t) = (8, 4), (160, 72), \ldots$$

(ii) $\frac{1}{2}(1 + \sqrt{5}) = 1 + \dfrac{2t}{c + t +} \left(\dfrac{c}{2c}\right)_\infty,$

where $c$ and $t$ satisfy $(2c + 1)^2 - 5(2t)^2 = 1$, so that

$$(2c + 1) + 2t\sqrt{5} = (2 + \sqrt{5})^{2n}, \qquad (n = 1, 2, \ldots,)$$

for example,

$$(c, t) = (4, 2), (80, 30), \ldots,$$

(iii) $\frac{1}{2}(1 + \sqrt{5}) = 1 + \dfrac{2t}{c + t +} \left(\dfrac{1}{2c}\right)_\infty,$

where $c$ and $t$ satisfy $c^2 - 5t^2 = -1$ and so

$$c + t\sqrt{5} = (2 + \sqrt{5})^{2n-1} \quad (n = 1, 2, \ldots,)$$

for example,

$$(c, t) = (2, 1), (38, 17) \ldots.$$

REFERENCES

1. J. D. Bankier and W. Leighton, *Numerical continued fractions*, Amer. J. Math., *64* (1942), 653–668.

*University of Malaya*

# PERTURBATION OF THE CONTINUOUS SPECTRUM OF EVEN ORDER DIFFERENTIAL OPERATORS

JOHN B. BUTLER, JR.

**1. Introduction.** Let $L_0$ be a differential operator of even order $n = 2\nu$ on the half open interval $0 \leqslant t < \infty$ which is formally self adjoint and satisfies the conditions of Kodaira **(5**, p. 503**)**. We consider a perturbed operator of the form $L_\epsilon = L_0 + \epsilon q$ where $q(t)$ is a real-valued bounded function and $\epsilon$ is a real parameter. The object of this paper is to set up conditions on the operator $L_0$ and the function $q(t)$ such that $L_\epsilon$ determines a self-adjoint operator $H_\epsilon$ and such that the spectral resolution operator $E^\epsilon(\Delta)$ corresponding to $H_\epsilon$ is analytic in a neighbourhood of $\epsilon = 0$, where $\Delta$ is a closed bounded interval.

Our conditions are a natural generalization of conditions considered by Moser for the case $n = 2$**(6)**. Moser has given a number of examples showing that when his conditions do not hold $E^\epsilon(\Delta)$ need not be analytic. However, Moser's conditions are not necessary. Brownell has demonstrated analyticity of $E^\epsilon(\Delta)$ for second order differential operators (in $E_n$) under conditions different from Moser's **(2)**.

Our main result is Theorem 4 which gives sufficient conditions that $E^\epsilon(\Delta)$ be analytic. Theorem 4 is an easy consequence of Theorem 3. The proof of Theorem 3 hinges upon the Neumann expansion for the resolvent kernel of the perturbed operator $H_\epsilon$ and on the behaviour of the resolvent kernel of the unperturbed operator $H_0$ under change of boundary conditions at $t = 0$. We discuss the former of these topics in § 4 and the latter in § 3. Section 2 is devoted to definitions and needed facts. The restrictions that we impose on $L_0$, $q$ are stated at the end of § 2.

The assumption that $q(t)$ is bounded can be removed. In § 6 we indicate briefly how this may be done.

The significance of analyticity of the spectral measure $E^\epsilon(\Delta')$ for $\Delta' \subset \Delta$, $\Delta$ a fixed bounded interval, is that it implies that points in the spectrum of $H_\epsilon$ which lie inside $\Delta$ remain fixed under the perturbation **(6; 7)**. Our assumptions imply that $\Delta$ contains only points of the continuous spectrum of $H_0$ (cf. assumption (ii)). Therefore, our results may be interpreted as sufficient conditions that the continuous spectrum remain fixed under perturbation.

The author wishes to thank F. H. Brownell for many helpful suggestions in the preparation of this paper.

Received March 4, 1959. This paper extends results presented to the American Mathematical Society, November 29, 1958, under the title *A note on perturbation theory of ordinary differential operators*.

**2. Basic definitions and assumptions.** We shall use the standard notation from the theory of ordinary differential operators **(3; 5)**. The notation $(u, v)$ will mean the inner product of two functions in $L_2(0, \infty)$. The norm of $u$ is $\|u\| = (u, u)^{\frac{1}{2}}$. Let $[u, v](t)$ be the bilinear form associated with the differential operator $L_0$ such that

$$\int_0^t (L_0 u\bar{v} - u\,\overline{L_0 v})\,dt = [u, v](t) - [u, v](0). \tag{2.1}$$

Since $t = 0$ is a regular point there exists a complete canonical set of boundary functions $\psi_{0j}(t)$ and regular solutions $s_j(t, \lambda)$ of $L_0 u = \lambda u$, $j = 1, \ldots, n$ such that

$$[\psi_{0j}, \psi_{0k}](0) = [\psi_{0j}, s_k](0) = [s_j, s_k](0) = \epsilon_{jk} \tag{2.2}$$

and $\epsilon_{jk} = +1$, $k = j + \nu$, $\epsilon_{jk} = -1$, $k = j - \nu$, $\epsilon_{jk} = 0$ otherwise **(4; 5**, p. 505**)**. We shall suppose the differential problem

$$L_0 u = \lambda u,\ [\psi_{0j}, u](0) = 0, \quad j = 1, \ldots, \nu \tag{2.3}$$

is self adjoint **(5**, p. 521**)**. In the case $n = 2$ this reduces to the limit point case at $t = \infty$.

Repeated indices will mean summation unless the contrary is explicitly stated. Latin indices are to be summed over $1, \ldots, n$ and Greek over $1, \ldots, \nu$.

Let $\mathscr{D}$ be the set of functions in $L_2(0, \infty)$ such that for $u \in \mathscr{D}$ we have $u^{(i)}(t) \in \mathscr{C}^1[0, \infty)$, $i = 1, \ldots, n - 1$, $u^{(n-1)}(t)$ is absolutely continuous in every closed subinterval of $[0, \infty)$, and $L_0 u \in L_2(0, \infty)$. Let $\mathscr{D}_\infty$ be the set of functions in $\mathscr{D}$ which vanish outside some closed bounded interval. The operator $L_0$ determines a self-adjoint operator $H_0$ as follows: We define $\mathscr{D}_{H_0}$ to be the set of functions

$$\mathscr{D}_{H_0} = \{u | u \in \mathscr{D} \quad \text{and} \quad [\psi_{0j}, u](0) = 0, j = 1, \ldots, \nu\}$$

and define $H_0 u = L_0 u$ for $u \in \mathscr{D}_{H_0}$ **(5**, p. 521**)**. Since we are assuming $q(t)$ bounded it follows at once that $L_\epsilon = L_0 + \epsilon q$ determines a self-adjoint operator $H_\epsilon$ with

$$\mathscr{D}_{H_\epsilon} = \mathscr{D}_{H_0}$$

and

$$H_\epsilon u = L_\epsilon u,\ u \in \mathscr{D}_{H_0}.$$

The assumption that the boundary value problem (2.3) is self-adjoint implies the following facts (which are all derived from **(5)**): There exist $\nu$ vectors $f_\beta(\lambda) = (f_\beta{}^1, \ldots, f_\beta{}^n)$, $\beta = \nu + 1, \ldots, n$ such that $w_\beta(t, \lambda) = f_\beta{}^j s_j$ are the eigenfunctions of $L_0 u = \lambda u$, $\mathscr{I}(\lambda) \neq 0$, $w_\beta(t, \lambda) \in L_2(0, \infty)$. Corresponding to the boundary conditions $[\psi_{0\iota}, u](0) = 0$ we may choose vectors $f_\alpha = (\delta_\alpha{}^1, \ldots, \delta_\alpha{}^n)$, $\alpha = 1, \ldots, \nu$. Then $w_\alpha = f_\alpha{}^j s_j$ satisfy $[\psi_{0j}, w_\alpha](0) = 0$, $j = 1, \ldots, \nu$, $\alpha = 1, \ldots, \nu$ by (2.2).

The Green's function corresponding to $H_0$ may be constructed as follows. Define the characteristic matrix $M^{ij}$ by

$$M_{ij} = \sum_{\alpha,\beta} F_{\alpha\beta} f_\beta{}^j f_\alpha{}^i$$

where $\alpha = 1, \ldots, \nu$, $\beta = \nu + 1, \ldots, n$ and $F_{\alpha\beta}$ is the inverse matrix of $[w_\alpha, w_\beta](t)$. The Green's function is by **(5**, p. 511**)**

$$G^0(t, \tau, \lambda) = M^{jk}(\lambda) s_j(t, \lambda) s_k(\tau, \lambda), \qquad t > \tau. \tag{2.4}$$

The spectral resolution operator $E^0(\Delta)$ corresponding to $H^0$ is defined in terms of the Green's function* by

$$E^0(\Delta)u = \frac{1}{2\pi i} \lim_{\delta \to 0+} \mathscr{I}\left\{\int_{\Gamma(\delta)} (G^0(t, \cdot, \lambda), \bar{u}) d\lambda\right\}, \; u \in \mathscr{D}_\infty \tag{2.5}$$

where $\Gamma(\delta)$ is the polygonal path connecting the points $\alpha + i\delta$, $\alpha + 2i\delta$, $\beta + 2i\delta$, $\beta + i\delta$, $\Delta = \{l | \alpha < l < \beta\}$. Formula (2.5) may be written **(5**, p. 528**)**

$$E^{(0)}(\Delta)u = \int_\Delta s_j(t, l)(s_k, \bar{u}) d\rho^{jk}(l) \qquad u \in \mathscr{D}_\infty \tag{2.6}$$

where

$$\rho^{jk}(\Delta) = \frac{1}{2\pi i} \lim_{\delta \to 0+} \mathscr{I}\left\{\int_{\Gamma(\delta)} M^{jk}(\lambda) d\lambda\right\}. \tag{2.7}$$

For two arbitrary $l$-measurable vector functions $\phi_i(l)$, $\psi_i(l)$, $i = 1, \ldots, n$ we have the inequality

$$\left|\int_{-\infty}^{\infty} \phi_j(l) \overline{\psi_k(l)} d\rho^{jk}(l)\right|^2 < \int_{-\infty}^{\infty} \phi_j \bar{\phi}_k d\rho^{jk}(l) \int_{-\infty}^{\infty} \psi_j \bar{\psi}_k d\rho^{jk}(l). \tag{2.8}$$

If $u \in \mathscr{D}_\infty$, $\phi_j = (s_j, u)$ then by **(5**, p. 537**)**

$$||u||^2 = \int_{-\infty}^{\infty} |\phi_j(l)|^2 d\rho^{jk}(l). \tag{2.9}$$

The following assumptions are basic for the theorems to be given below. We shall require† that $L_0$ and $q$ are such that, for $l$ in a fixed finite interval $\Delta$,

(i) $\int_0^\infty \Phi^2(t) |q(t)| dt < \gamma < \infty$

where $\Phi(t) = \sup |s_j(t, \lambda)|$, $j = 1, \ldots, n$, $l \in \Delta$, $0 < \delta < \delta_0$, $\lambda = l + i\delta$.

(ii) $\lim_{\delta \to 0+} |M^{jk}(l + i\delta)| < K$

for $l \in \Delta$, $0 < \delta < \delta_0$, $j, k = 1, \ldots, n$.

---

*We assume the end points of $\Delta$ are not in the point spectrum.

†This assumption is weakened in § 6.

(iii) for all vector functions $\phi^i(l)$ defined on $\Delta$,*

$$\phi^j(l)\overline{\phi^k(l)}\rho^{jk}(\Delta') - \phi^j(l)\overline{\phi^j(l)}\rho^{jj}(\Delta') > 0 \tag{2.10}$$

for $l \in \Delta' \subset \Delta$.

(iv) if $s_{j+p}'$ are permutations of the regular solutions $s_j$ according to the rules $s_{j+p}' = s_{j+p}$ for $j + p < n$ and $s_{j+p}' = s_{j+p-n}$ for $j + p > n$, then for $p = 1, \ldots, n$

$$\int_\Delta s'_{j+p}s'_{k+p}d\rho^{jk}(l) \tag{2.11}$$

is the kernel of a bounded operator with bound $P^2$.

The assumptions (i) and (ii) reduce to ones considered by Moser for the case $n = 2$ (**6**, pp. 367, 388). Assumption (i) asserts roughly that the operator $q$. is relatively bounded with respect to $L_0$. Assumption (ii) implies that $M^{jk}$ does not have any poles in $\Delta$ so that $\Delta$ contains only continuous spectrum. Assumptions (iii) and (iv) are unnecessary in the case $n = 2$ as they are automatically satisfied. Assumption (iii) is a definiteness condition on the bilinear form associated with the matrix $\rho^{jk}(\Delta')$. This condition is trivially satisfied if $\rho^{jk}(\Delta')$ is diagonal and for that reason holds when $n = 2$. Assumption (iv) is the key assumption upon which our proof of Theorem 4 depends. The fact that (iv) holds when $n = 2$ is also used by Moser in his paper (**6**, p. 382). In § 3 we shall discuss the meaning of assumption (iv) and show that it is valid for a broad class of operators $L_0$.

**3. Changes in boundary conditions at $t = 0$.** In this section we shall study kernels

$$\int_\Delta s_j(t, l)s_k(\tau, l)d\tilde{\rho}^{jk}(l)$$

corresponding to self-adjoint boundary value problems of the form

$$L_0u = \lambda u, \; [\tilde{\psi}_{0j}, u](0) = 0 \qquad j = 1, \ldots, \nu \tag{3.1}$$

where the functions $\tilde{\psi}_{0j}$ are linear combinations of $\psi_{0j}$. The object of this section is to show that, under certain restrictions on $L_0$, and by appropriate choice of the boundary functions $\tilde{\psi}_{0i}$, that the kernels (2.11) of assumption (iv) may be written in terms of the kernel

$$\int_\Delta s_j(t, l)s_k(\tau, l)d\tilde{\rho}^{jk}(l).$$

Therefore we will have a means of testing when assumption (iv) holds. The theorem is the following:

*Ibid.

THEOREM 1. *If $L_0$ is a differential operator satisfying assumption* (ii) *and if the functions $f_\beta{}^j(\lambda)$ corresponding to $L_0$ satisfy the property that for $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_0$, the determinants of the $(\nu \times \nu)$ minors of the matrix*

$$\text{(3.2)} \qquad \begin{pmatrix} f^1_{\nu+1}(\lambda) \dots f^n_{\nu+1}(\lambda) \\ \dots \\ f^1_{\nu+\nu}(\lambda) \dots f^n_{\nu+\nu}(\lambda) \end{pmatrix}$$

*have moduli greater than $k_1$ and less than $k_2$, $0 < k_1 < k_2$, and the difference of the arguments $\alpha$ of any two $(\nu \times \nu)$ minors lies in a sector such that $0 < \theta < \alpha < \pi - \theta < \pi$, $\sin \theta > k_1$, then for some function $a_{jk}(l)$*

$$\text{(3.3)} \qquad \int_\Delta s'_{j+p}(t, l) s'_{k+p}(\tau, l) d\rho^{jk}(l) = \int_\Delta s_j(t, l) s_k(\tau, l) a_{ij}(l) d\bar{\rho}^{jk}(l)$$

*where $a_{jk}(l)$ are uniformly bounded and $\bar{\rho}^{jk}(\Delta)$ is the spectral density matrix corresponding to a self-adjoint problem $L_0 u = \lambda u$, $[\bar{\psi}_{0j}, u](0) = 0, j = 1, \dots, \nu$.*

*Proof.* First we introduce the notation $j_p, j_p', j_p''$ for permutations of $j = 1, \dots, n$ defined by:

$$\begin{cases} j_p = j + p, j + p \leqslant n, j_p = j + p - n, j + p > n \\ j'_p = j - p, j \geqslant p + 1, j'_p = n + j - p, j \leqslant p \\ j''_p = j + p + \nu, j + p \leqslant \nu, j''_p = j + p - \nu, j + p > \nu. \end{cases}$$

Define

$$\bar{\psi}_{0j} = \delta^k_{j_p} \psi_{0k}, \quad j = 1, \dots, \nu.$$

Using (2.2) we get

$$\text{(3.4)} \qquad [\bar{\psi}_{0j}, \bar{\psi}_{0k}](0) = 0, \qquad j, k = 1, \dots, \nu.$$

Formula (3.4) shows that the problem (3.1) is self adjoint when $\bar{\psi}_{0j} = \delta_{jp}{}^k \psi_{0k}$. Let $\bar{M}^{jk}(\lambda)$ be the characteristic matrix corresponding to (3.1). Then $\bar{M}^{jk}(\lambda)$ can be explicitly constructed (cf. § 2) as follows:

$$\text{(3.5)} \qquad \bar{M}^{jk}(\lambda) = \sum_{\alpha, \beta} \bar{F}_{\alpha\beta} \bar{f}_\beta{}^j \bar{f}_\alpha{}^k, \quad \alpha = 1, \dots, \nu, \beta = \nu + 1, \dots, n$$

where

$$\bar{f}_\alpha{}^j = \delta^j_{\alpha_p}, \quad \alpha = 1, \dots, \nu$$

and $\bar{f}_\beta{}^j = f_\beta{}^j, \beta = \nu + 1, \dots, n$, $\bar{F}_{\alpha\beta}$ is the inverse of $[\bar{w}_\alpha, \bar{w}_\beta](t)$, $\bar{w}_\alpha = \bar{f}_\alpha{}^j s_j(t, \lambda)$, $\bar{w}_\beta = \bar{f}_\beta{}^j s_j(t, \lambda)$. Using (2.2) we have

$$\text{(3.6)} \qquad [\bar{\omega}_\alpha, \bar{\omega}_\beta] = \sum_{j=1}^{\nu} \delta^j_{\alpha_p} f^{j+\nu}_\beta - \delta^{j+\nu}_{\alpha_p} f^j_\beta = \begin{cases} f_\beta^{\nu+\alpha_p}, & \alpha_p \leqslant \nu \\ -f_\beta^{\alpha_p-\nu}, & \alpha_p > \nu \end{cases}$$

By (3.5), (3.6) $\bar{M}^{jk}(\lambda)$ may be written*

*The sign is positive if $k \leqslant \nu$ and negative if $k > \nu$.

$$\tilde{M}^{jk}(\lambda) = (\pm) \det \tilde{A}(j, k)/\det \tilde{B}, \qquad k = 1_p, 2_p, \ldots, \nu_p \tag{3.7}$$

where $\tilde{B}$ is the matrix

$$\tilde{B} = \begin{pmatrix} f_{\nu+1}^{1_p''}(\lambda) \ldots f_{\nu+1}^{\nu_p''}(\lambda) \\ \ldots \\ f_{\nu+\nu}^{1_p''}(\lambda) \ldots f_{\nu+\nu}^{\nu_p''}(\lambda) \end{pmatrix} \tag{3.8}$$

and $\tilde{A}(j, k)$ is the matrix obtained from $\tilde{B}$ by replacing the elements of the $k_p$'th column with the terms $f_{\nu+1}{}^j, f_{\nu+2}{}^j, \ldots, f_{\nu+\nu}{}^j$. The hypothesis of the theorem implies that for $j, k = 1_p, 2_p, \ldots, \nu_p$, $K_1 < \det |\tilde{A}(j, k)| < k_2$.

Now that $\tilde{M}^{jk}$ has been constructed the remainder of the proof consists in demonstrating that (3.3) holds for some $a_{jk}(l)$. By the definition of $j_p'$ we may write

$$s'_{j+p}(t, l)s'_{k+p}(\tau, l)\mathscr{I}\{M^{jk}(\lambda)\} = s_j(t, l)s_k(\tau, l)\mathscr{I}\{M^{j'pk'p}(\lambda)\}. \tag{3.10}$$

(Note that

$$\mathscr{I}\{M^{j'pk'p}(\lambda)\} = 0, \quad j, k \neq 1_p, 2_p, \ldots, \nu_p.)$$

Now define $a_{jk}(\lambda)$ by the equation

$$a_{jk}(\lambda) = \begin{cases} \mathscr{I}\{M^{j'pk'p}(\lambda)\}/\{\mathscr{I} M^{jk}(\lambda)\}, & j, k = 1_p, 2_p, \ldots, \nu_p \\ 0, \text{ otherwise.} \end{cases} \tag{3.11}$$

Since $\tilde{M}^{jk} = \pm \det \tilde{A}(j, k)/\det \tilde{B}$ we have by (ii), (3.9)

$$|a_{jk}(\lambda)| < Kk_2/k_1 \sin\theta < Kk_2/k_1^2 \tag{3.12}$$

so that $a_{jk}(\lambda)$ are uniformly bounded, $l \in \Delta$, $0 < \delta < \delta_0$. By using (2.7), (3.11) and the theorem of Helly-Bray (**8**, p. 164) it follows that for $\Delta' \subset \Delta$

$$\begin{aligned} \rho^{j'pk'p}(\Delta') &= \lim_{\delta \to 0+} \frac{1}{\pi} \int_{\Delta'} \mathscr{I}\{M^{j'pk'p}(\lambda)\}\, dl \\ &= \lim_{\delta \to 0+} \frac{1}{\pi} \int_{\Delta'} a_{jk}(\lambda)\mathscr{I}\{M^{jk}(\lambda)\}\, dl. \end{aligned} \tag{3.13}$$

From (3.12), (3.13) we have

$$|\rho^{j'pk'p}(\Delta')| < K\, k_2/k_1^2 \quad (\text{variation } \rho^{jk}(\Delta')),\ \Delta' \subset \Delta. \tag{3.14}$$

The inequality (3.14) implies that functions $a_{ij}(l)$ exist (**8**, p. 215) such that

$$\rho^{j'pk'p}(\Delta') = \int_{\Delta'} a_{ij}(l)d\rho^{jk}(l),\ \Delta' \subset \Delta. \tag{3.15}$$

By (3.10), (3.15) we obtain (3.3).

Theorem 1 leads to a sufficient condition that assumption (iv) should hold; if the hypothesis of Theorem 1 is satisfied and if

$$\int_{\Delta} s_j(t, l)s_k(\tau, l)a_{jk}(l)d\bar{\rho}^{jk}(l)$$

is the kernel of a bounded operator then assumption (iv) holds. One can easily show by direct calculation that in the case $n = 2$ the hypothesis of Theorem 1 is satisfied and

$$\int_\Delta s_j(t, l)s_k(\tau, l)a_{jk}(l)d\bar{\rho}^{jk}(l)$$

is the kernel of a bounded operator. Therefore, assumption (iv) holds automatically when $n = 2$ (**6**, p. 382).

**4. Neumann series for the resolvent.** Following (**1**, p. 560) we define functions $G^{(\nu)}(t, \tau, \lambda)$ by setting $G^{(0)} = G^{(0)}$ and

$$G^{(\nu)} = [+ G^{(\nu-1)}q]\cdot[G^{(0)}] = [+ G^0q]^\nu\cdot[G^0], \quad \nu = 1, 2, \ldots, \tag{4.1}$$

where the brackets indicate integration as follows

$$[G^0q]\cdot[G^0] = \int_0^\infty G^0(t, \xi, \lambda)q(\xi)G^0(\xi, \tau, \lambda)\, d\xi.$$

The object of this section is to show that $G^\epsilon = \sum(-\epsilon)^\nu G^{(\nu)}$ is the kernel of the resolvent of the operator $H_\epsilon$.

LEMMA 1. *If $G^{(\nu)}$ is defined by* (4.1) *and assumptions* (i) *and* (ii) *hold, then for $|\epsilon| < (\gamma K n^2)^{-1}$, $l \in \Delta$, $0 < \delta < \delta_0$ the series $G^\epsilon = \sum(-\epsilon)^\nu G^{(\nu)}$ converges uniformly and absolutely and*

$$|G^{(\nu)}| < \Phi(t)\Phi(t)\gamma^\nu(Kn^2)^{\nu H}, \; \nu = 0, 1, 2 \ldots. \tag{4.2}$$

*Proof.* The inequality (4.2) holds for $\nu = 0$ by assumption (ii) and (2.4). Suppose (4.2) true for $(\nu - 1)$. Then by (4.1)

$$G^{(\nu)} = + \int_0^\infty G^{(0)}(t, \xi, \lambda)q(\xi)G^{(\nu-1)}(\xi, \tau, \lambda)\, d\xi. \tag{4.3}$$

Using assumptions (i), (ii), and (2.4) we get

$$\begin{aligned} |G^{(\nu)}| &< \Big| \sum M^{jk}(\lambda)\Big\{s_j(t, \lambda)\int_0^t s_k(\xi, \lambda)q(\xi)G^{(\nu-1)}(\xi, \tau, \lambda)\, d\xi \\ &\qquad + s_k(t, \lambda)\int_t^\infty s_j(\xi, \lambda)q(\xi)G^{(\nu-1)}(\xi, \tau, \lambda)\, d\xi\Big\}\Big| \\ &< Kn^2\Phi(t)\int_0^\infty \Phi(\xi)|q(\xi)|\gamma^{\nu-1}\Phi(\xi)\Phi(\tau)(Kn^2)^\nu\, d\xi \\ &< (Kn^2)^{\nu+1}\gamma^{\nu-1}\Phi(t)\Phi(\tau)\int_0^\infty \Phi^2(\xi)|q(\xi)|\, d\xi \\ &< (Kn^2)^{\nu+1}\gamma^\nu\Phi(t)\Phi(\tau). \end{aligned} \tag{4.4}$$

This proves (4.2). The absolute convergence of the series for $G^\epsilon$ follows from (4.2). We also need the following lemma:

LEMMA 2. *If*

$$\mathscr{G}^{(\nu)}(\lambda)u = \int_0^\infty G^{(\nu)}u\, d\tau$$

*where* $G^{(\nu)}$ *is defined by* (4.1) *and if assumptions* (i) *and* (ii) *hold, then* $\mathscr{G}^{(\nu)}(\lambda)$ *is a bounded operator and*

$$\| \, |q|^{\frac{1}{2}}\mathscr{G}^{(\nu)}u\| < (\gamma K n^2)^\nu \frac{\max |q|^{\frac{1}{2}}}{\mathscr{I}(\lambda)} \|u\| \qquad \nu = 0, 1, 2, \ldots, \tag{4.5}$$

*Proof.* For $\nu = 0$

$$\| \, |q|^{\frac{1}{2}}\mathscr{G}^{(0)}(\lambda)u\| < \max |q|^{\frac{1}{2}} |||\mathscr{G}^{(0)}(\lambda)||| \, \|u\| < \max |q|^{\frac{1}{2}} \frac{1}{\mathscr{I}(\lambda)} \|u\|.$$

Suppose (4.5) true for $(v - 1)$. Then using (2.4) and assumptions (i) and (ii),

$$\begin{aligned} |\, |q|^{\frac{1}{2}}\mathscr{G}^{(\nu)}(\lambda)u| &< \sum |M^{jk}| \Big\{ |q(t)|^{\frac{1}{2}} |s_j(t, \lambda)| \int_0^t |s_k(\xi, \lambda) q(\xi) \mathscr{G}^{(\nu-1)}u| d\xi \\ &\quad + |q(t)|^{\frac{1}{2}} |s_k(t, \lambda)| \int_t^\infty |s_j(\xi, \lambda) q(\xi) \mathscr{G}^{(\nu-1)}u| d\xi \Big\} \\ &< (Kn^2) |q(t)|^{\frac{1}{2}} \Phi(t) \int_0^\infty \Phi(\xi) |q(\xi) \mathscr{G}^{(\nu-1)}u| d\xi. \end{aligned} \tag{4.6}$$

From (4.6) it follows

$$\begin{aligned} \| \, |\cdot q|^{\frac{1}{2}}\mathscr{G}^{(\nu)}u\|^2 &= \int_0^\infty |\, |q|^{\frac{1}{2}}\mathscr{G}^{(\nu)}u|^2 dt \\ &< (Kn^2)^2 \int_0^\infty \Phi^2(t) |q(t)| dt \int_0^\infty \Phi^2(\xi) |q(\xi)| d\xi \int_0^\infty |q(\xi)| |\mathscr{G}^{(\nu-1)}u|^2 d\xi \\ &< (Kn^2)^2 \gamma^2 (\gamma K n^2)^{2\nu-2} \left\{ \frac{\max |q|^{\frac{1}{2}}}{\mathscr{I}(\lambda)} \right\}^2 \|u\|^2 \\ &< (\gamma K n^2)^{2\nu} \left\{ \frac{\max |q|^{\frac{1}{2}}}{\mathscr{I}(\lambda)} \right\}^2 \|u\|^2. \end{aligned} \tag{4.7}$$

Lemma 1 and Lemma 2 imply:

THEOREM 2. *If* $G^{(\nu)}$ *is defined by* (4.1) *and assumptions* (i) *and* (ii) *hold, then for* $|\epsilon| < (\gamma K n^2)^{-1}$, $l \in \Delta$, $0 < \delta < \delta_0$ *the series* $G^\epsilon = \sum (-\epsilon)^\nu G^{(\nu)}$ *represents the kernel of the resolvent* $R^\epsilon(\lambda) = (H_\epsilon - \lambda 1)^{-1}$ *of the operator* $H_\epsilon$.

*Proof.* Let

$$\mathscr{B}^\epsilon(\lambda) = 1 + (+q) \sum_{\nu=0}^{\infty} (-\epsilon)^{\nu+1} \mathscr{G}^{(\nu)}(\lambda).$$

By Lemma 2 the series for $\mathscr{B}^\epsilon(\lambda)$ converges uniformly in norm for $|\epsilon| < (\gamma K n^2)^{-1}$ and defines a bounded operator. Since $\mathscr{G}^\epsilon(\lambda) = \mathscr{G}^{(0)}(\lambda)\mathscr{B}^\epsilon(\lambda)$ and both $\mathscr{G}^{(0)}(\lambda)$,

$\mathscr{B}^\epsilon(\lambda)$ are bounded operators it follows $\mathscr{G}^\epsilon(\lambda)$ is a bounded operator. In order to show that $\mathscr{G}^\epsilon(\lambda)$ is the resolvent it is sufficient to show the range of $\mathscr{G}^\epsilon(\lambda)$ is in $\mathscr{D}_{H_0}$ and

$$(4.8) \qquad (L_\epsilon - \lambda 1)\mathscr{G}^\epsilon(\lambda)u = u, \qquad u \in L_2(0,\infty)$$

$$(4.9) \qquad \mathscr{G}^{(\epsilon)}(\lambda)(L_\epsilon - \lambda 1)u = u, \quad u \in \mathscr{D}_{H_0}.$$

Since the range of $\mathscr{G}^0(\lambda)$ is $\mathscr{D}_{H_0}$ and since $\mathscr{B}^\epsilon(\lambda)$ is bounded it follows the range $\mathscr{G}^\epsilon(\lambda)$ is contained in $\mathscr{D}_{H_0}$. Formula (4.8) can be proved by direct calculation using the definition of $G^{(\nu)}$ and Lemmas 1 and 2 (we shall omit the computation as it is standard (**1**, p. 562)). To prove (4′9) set

$$w = u - \mathscr{G}^\epsilon(\lambda)(L_\epsilon - \lambda 1)u, \quad u \in \mathscr{D}_{H_0}.$$

Since $w$ is the difference of two elements of $\mathscr{D}_{H_0}$ it follows

$$w \in \mathscr{D}_{H_0}.$$

Then

$$(H_\epsilon - \lambda 1)w = (L_\epsilon - \lambda 1)w = (L_\epsilon - \lambda 1)u - (L_\epsilon - \lambda 1)\mathscr{G}^\epsilon(\lambda)(L_\epsilon - \lambda 1)u = 0.$$

This implies $w = (H_\epsilon - \lambda 1)^{-1}\, 0 = 0$.

For later use define the modified resolvent kernels $\bar{G}^{(\nu)}(t, \tau, \lambda)$ by

$$(4.5) \qquad \bar{G}^{(0)} = M^{jk}(l + i\delta)s_j(t, l)\, s_k(\tau, l), \qquad t > \tau$$

$$(4.6) \qquad \bar{G}^{(\nu)} = [\bar{G}^{(0)}q]^\nu \,.\, [\bar{G}^0] \qquad \nu = 1, 2 \ldots .$$

Since $s_j(t, \lambda)$ are entire in $\lambda$ the functions $\bar{G}^{(\nu)}$ have the same type of singularities along the real axis as $G^{(\nu)}$. Also $\bar{G}^{(\nu)}$ satisfy Lemmas 1 and 2.

**5. Analyticity of $E^\epsilon(\Delta)$.** In this section we show that the spectral measure $E^\epsilon(\Delta)$ corresponding to $H_\epsilon$ is an analytic operator in a neighbourhood of $\epsilon = 0$. Define the function $\mathscr{E}^{(\nu)}(t, \tau)$ by

$$(5.1) \qquad \mathscr{E}^{(\nu)} = \lim_{\delta \to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} G^{(\nu)} d\lambda\right\}.$$

We shall show that $\mathscr{E}^{(\nu)}$ are kernels of bounded operators $E^{(\nu)}(\Delta)$ and that $E^{(\epsilon)}(\Delta) = \sum \epsilon^\nu E^{(\nu)}(\Delta)$ for sufficiently small $\epsilon$. To do this first consider the approximate kernel $\hat{\mathscr{E}}^{(\nu)}$ defined by

$$(5.2) \qquad \hat{\mathscr{E}}^{(\nu)} = \lim_{\delta \to 0+} \frac{1}{\pi} \int_\Delta \mathscr{I}(\bar{G}^{(\nu)})\, dl$$

where $\bar{G}^{(\nu)}$ is defined by (4.6). We shall first prove that $\hat{\mathscr{E}}^{(\nu)} = \mathscr{E}^{(\nu)*}$:

LEMMA 3. *If $\mathscr{E}^{(\nu)}(\Delta)$, $\hat{\mathscr{E}}^{(\nu)}(\Delta)$ are defined by* (5.1) *and* (5.2) *and if assumptions* (i), (ii), *and* (iii) *hold, then $\mathscr{E}^{(\nu)}(\Delta) = \hat{\mathscr{E}}^{(\nu)}(\Delta)$.*

---

*The existence of $\hat{\mathscr{E}}^{(\nu)}$ is insured by (4.5), (4.6), and (ii) cf. (**9**, p. 346, **22.23**).

*Proof.* By a routine calculation which will be omitted one can show using (ii), (2.4), (4.1), and (4.6) that for $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_0$,

$$|G^{(\nu)}(t, \tau, \lambda) - \bar{G}^{(\nu)}(t, \tau, \lambda)| < M_1\delta, \tag{5.3}$$

where $M_1$ depends on $(t, \tau)$ but is independent of $\lambda$. Using (5.3) we have

$$\mathscr{E}^{(\nu)}(\Delta) = \lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} G^{(\nu)} d\lambda\right\} = \lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} \bar{G}^{(\nu)} d\lambda\right\}. \tag{5.4}$$

Next (4.2) implies

$$\lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} \bar{G}^{(\nu)} d\lambda\right\} = \lim_{\delta\to 0+} \frac{1}{\pi} \int_{\Delta} \mathscr{I}(\bar{G}^{(\nu)}) dl = \hat{\mathscr{E}}^{(\nu)}. \tag{5.5}$$

By (5.4) and (5.5) $\mathscr{E}^{(\nu)}(\Delta) = \hat{\mathscr{E}}^{(\nu)}(\Delta)$.

THEOREM 3. *If $\hat{\mathscr{E}}^{(\nu)}(\Delta)$ is defined by* (5.2) *and if assumptions* (i), (ii), (iii), *and* (iv) *hold then $\hat{\mathscr{E}}^{(\nu)}(\Delta)$ is the kernel of a bounded operator $E^{(\nu)}(\Delta)$ and*

$$|(E^{(\nu)}(\Delta)u, v)| < p^2(4\nu)(\gamma K n^2)^\nu n^3 \|u\| \, \|v\| \qquad u, v \in L_2(0, \infty). \tag{5.6}$$

*Proof.* From the definition of $\bar{G}^{(\nu)}$ one can show by induction that

$$\mathscr{I}(\bar{G}^{(\nu)}) = \sum_{\mu+\chi=\nu} [\overline{\bar{G}^{(\nu)}} q]^\mu \cdot \mathscr{I}(\bar{G}^0) \cdot [q\bar{G}^{(0)}]^\chi. \tag{5.7}$$

Next by (2.4) and (4.5) $\mathscr{I}(\bar{G}^0) = \mathscr{I}(M^{jk}) s_j(t, l) s_k(\tau, l)$, $t > \tau$ and (5.7) may be written

$$\mathscr{I}(\bar{G}^{(\nu)}) = \sum_{\mu+\chi=\nu} \bar{H}_j^{(\mu)}(t) H_k^{(\chi)}(\tau) \mathscr{I}(M^{jk}) \tag{5.8}$$

where

$$\begin{aligned} H_j^{(\mu)}(t) = \sum_{p,m} M^{pm} s_p(t, l) \int_0^t d\xi_1 \int_0^\infty s_m(\xi_1, l) q(\xi_1) \\ \bar{G}^{(\mu-2)}(\xi_1, \xi_2, \lambda) q(\xi_2) s_j(\xi_2, l) d\xi_2 \\ + M^{mp} s_p(t, l) \int_t^\infty d\xi_1 \int_0^\infty s_m(\xi_1, l) q(\xi_1) \bar{G}^{(\mu-2)}(\xi_1, \xi_2, \lambda) q(\xi_2) s_j(\xi_2, l) d\xi_2 \\ = \sum_p s_p(t, l) \left\{\int_0^t \eta_{j,p}^{(\mu)}(\xi, \lambda) d\xi + \int_t^\infty \zeta_j^{(\mu)}(\xi, \lambda) d\xi\right\}. \end{aligned} \tag{5.9}$$

The integrals in (5.9) converge absolutely and may be estimated using (4.2). Define for fixed values of $j$, $p$, $\mu$ (no summation)

$${}_1Q_{p,\mu}^j(t, \lambda) = s_p(t, l) \int_0^t \eta_{j,p}^{(\mu)}(\xi, \lambda) d\xi \tag{5.10}$$

$${}_2Q_{p,\mu}^j(t, \lambda) = s_p(t, l) \int_t^\infty \zeta_{j,p}^{(\mu)}(\xi, \lambda) d\xi. \tag{5.11}$$

Using (ii) and (4.2) it is easily seen that

$$\left|\int_0^t \eta_{j,p}^{(\mu)}(\xi,\lambda)d\xi\right| < (\gamma K n^2)^\mu \tag{5.12}$$

$$\left|\int_t^\infty \zeta_{j,p}^{(\mu)}(\xi,\lambda)d\xi\right| < (\gamma K n^2)^\mu.$$

With the notation $j_p$ introduced in Theorem 1 equation (5.8) becomes

$$\mathscr{I}(\bar{G}^{(\nu)}) = \sum_{\mu+\chi=\nu}\ \sum_{i_1,i_2=1} ({}_{i_1}\bar{Q}^j_{p,\mu})({}_{i_2}Q^k_{r,\chi})\mathscr{I}(M^{jk}) \tag{5.13}$$

$$= \sum_{\mu+\chi=\nu}\ \sum_{i_1 i_2=1}^{2} ({}_{i_1}\bar{Q}^j_{j_p,\mu})({}_{i_2}Q^k_{k_r,\chi})\mathscr{I}(M^{jk}).$$

When (5.13) is inserted in (5.2) and operations of limit and integration are interchanged we get for $u, v \in \mathscr{D}_\infty$

$$(E^{(\nu)}(\Delta)u,v) = \sum_{\mu+\chi=\nu}\ \sum_{i_1,i_2=1}^{2} \lim_{\delta\to0+}\frac{1}{\pi}\int_\Delta ({}_{i_1}\bar{Q}^j_{j_p\mu}, v)({}_{i_2}Q^k_{k_r,\chi}, \bar{u})\,\mathscr{I}(M^{jk}(\lambda)dl. \tag{5.14}$$

The interchange of limit operations in (5.14) is justified since the integrand is less than an absolutely integrable function (the integrand is less than $\Phi(t)|v(t)|\Phi(\tau)|u(\tau)|(\gamma K n^2)^\nu 2K$ by (5.12) and (ii) and this function is integrable for $u, v \in \mathscr{D}_\infty$). The remainder of the proof consists in estimating the term of (5.14). For $p$, $r$, $\mu$, $i_1$, $i_2$ fixed (no summation) we have by the Schwarz inequality

$$\left|\int_\Delta ({}_{i_1}\bar{Q}^j_{j_p,\mu}, v)({}_{i_2}Q^k_{k_r,\chi}, \bar{u})\mathscr{I}(M^{jk}(\lambda)dl\right|^2 \tag{5.15}$$

$$< \int_\Delta ({}_{i_1}\bar{Q}^j_{j_p\mu}, v)({}_{i_1}Q^k_{k_p\mu}, \bar{v})\mathscr{I}(M^{jk}(\lambda))dl$$

$$\times \int_\Delta ({}_{i_2}Q^j_{j_r,\chi}, \bar{u})({}_{i_2}\bar{Q}^k_{k_r,\chi}, u)\mathscr{I}(M^{jk}(\lambda))dl$$

since $\mathscr{I}(M^{jk}(\lambda))$ is a non-negative matrix (cf (**5**, p. 534)).

Again since $\mathscr{I}(M^{jk})$ is a non-negative matrix $|\mathscr{I}(M^{jk}(\lambda))| < (\mathscr{I}(M^{jj}))^{\frac{1}{2}}$ $(\mathscr{I}(M^{kk}))^{\frac{1}{2}}$, and we have

$$\left|\int_\Delta ({}_{i_1}Q^j_{j_p,\mu}, \bar{v})({}_{i_1}\bar{Q}^k_{k_p\mu}, v)\mathscr{I}(M^{jk}(\lambda))dl\right| \tag{5.16}$$

$$< \int_\Delta |({}_{i_1}Q^j_{j_p,\mu}, \bar{v})||({}_{i_1}\bar{Q}^k_{k_p,\mu}, v)|(\mathscr{I}(M^{jj}))^{\frac{1}{2}}(\mathscr{I}(M^{kk}))^{\frac{1}{2}}dl$$

$$< \frac{n}{2}\left(\int_\Delta |({}_{i_1}Q^j_{j_p,\mu}, \bar{v})|^2\mathscr{I}(M^{jj})dl + \int_\Delta |({}_{i_1}\bar{Q}^k_{k_p,\mu}, v)|^2\mathscr{I}(M^{kk})dl\right)$$

$$< n\int_\Delta |{}_{i_1}Q^j_{j_p,\mu}, \bar{v})|^2\mathscr{I}(M^{jj}(\lambda))dl.$$

By (5.10) and the Schwarz inequality

$$(5.17)\quad |({}_1Q^j_{j_p,\mu}, v)|^2 = \left|\int_0^\infty \eta_j^{(\mu)}(t,\lambda)dt \int_t^\infty s_{j_p}(\xi, l)v(\xi)d\xi\right|^2$$

$$< \int_0^\infty |\eta^{(\mu)}(t)|dt \int_0^\infty |\eta^{(\mu)}(t)| \left|\int_t^\infty s_{j_p}(\xi, l)v(\xi)d\xi\right|^2 dt$$

where $|\eta^{(\mu)}(t)| = \sup_{j\lambda_p} |\eta^\mu_{j,p}|$. Now apply assumptions (iii) and (iv), and (5.12), (5.16), and (5.17) to obtain

$$(5.18)\quad \lim_{\delta\to 0+} \frac{1}{\pi}\left|\int_\Delta ({}_1Q^j_{j_p,\mu}, \bar v)({}_1\bar Q^k_{k_p,\mu}, v)\mathscr{I}(M^{jk}(\lambda))dl\right|$$

$$< (\gamma Kn^2)^\mu \int_0^\infty |\eta^{(\mu)}(t)| \lim_{\delta\to 0+} \frac{1}{\pi}\int_\Delta \left|\int_t^\infty s_{pj}(\xi, l)v(\xi)d\xi\right|^2 \mathscr{I}(M^{jj}(\lambda)dl\, dt$$

$$= (\gamma Kn^2)^\mu \int_0^\infty |\eta^{(\mu)}(t)| \int_\Delta \left|\int_t^\infty s_{pj}(\xi, l)v(\xi)d\xi\right|^2 d\rho^{jj}(l)dt$$

$$< (\gamma Kn^2)^\mu \int_0^\infty |\eta^{(\mu)}(t)| \int_\Delta \left(\int_t^\infty s_{pj}(\xi, l)v(\xi)d\xi\right)\left(\int_t^\infty s_{pk}(\xi, l)v(\xi)\xi\right) d\rho^{jk}(l)dt$$

$$< n(\gamma Kn^2)^{2\mu}p^2||v||^2, \qquad v \in \mathscr{D}_\infty.$$

The identity

$$\lim_{\delta\to 0+} \frac{1}{\pi}\int_\Delta \left|\int_t^\infty s_{pj}(\xi, l)v(\xi)d\xi\right|^2 \mathscr{I}(M^{jk}(\lambda))dl = \int_\Delta \left|\int_t^\infty s_{pj}(\xi, l)v(\xi)d\xi\right|^2 d\rho^{jj}(l)$$

is by the theorem of Helly-Bray (**8**, pp. 163, 209). In exactly the same manner as (5.18) was obtained we get

$$(5.19)\qquad \lim_{\delta\to 0+} \int_\Delta ({}_iQ^j_{j,\mu}, \bar v)({}_i\bar Q^j_{jp\mu}, v)\mathscr{I}(M^{jk})dl < (\gamma Kn^2)^{2\mu}p^2||v||^2 n$$

$$(5.20)\qquad \lim_{\delta\to 0+\Delta} ({}_iQ^k_{kr,\chi}, u)({}_i\bar Q^k_{kr,\chi}, \bar u)\mathscr{I}(M^{jk})dl < (\gamma Kn^2)^{2\chi}p^2||u||^2 n$$

$$i = 1, 2.$$

Using (5.14), (5.14), (5.19), and (5.20) we have

$$(5.21)\qquad |(\mathrm{E}^{(\nu)}(\Delta)u, v)| < \nu P^2(\gamma Kn^2)^\nu(4n^3)||u||\,||v||, \qquad u, v \in \mathscr{D}_\infty.$$

The inequality (5.20) must hold for all $u$, $v$ in $L_2(0, \infty)$ and $E^{(\nu)}(\Delta)$ determines a bounded operator by a theorem by Frechet (**6**, p. 385).

Now we shall state our main theorem:

THEOREM 3. *If $L_\epsilon = L_0 + \epsilon q$ is a differential operator such that the problem $L_0 u = \lambda u$, $[\psi_{0j}, u](0) = 0$, $j = 1, \ldots, \nu$ is self adjoint and satisfies conditions* (i), (ii), (iii), *and* (iv) *then for $|\epsilon| < (\gamma Kn^2)^{-1}L_\epsilon$ determines a self-adjoint operator $H_\epsilon$ and the spectral measure $E^\epsilon(\Delta)$ corresponding to $H_\epsilon$ is an analytic operator.*

*Proof*. For $|\epsilon| < (\gamma K n^2)^{-1}$ we have the equalities

$$(5.22)\qquad \sum \epsilon^{\nu}(E^{(\nu)}(\Delta)u, v) = \sum (-\epsilon)^{\nu} \lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} (\mathscr{G}^{(\nu)}u, v)d\lambda\right\}$$
$$= \lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} \sum (-\epsilon)^{\nu}(\mathscr{G}^{(\nu)}u, v)d\lambda\right\}$$
$$= \lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} (\mathscr{G}^{\epsilon}u, v)d\lambda\right\}$$
$$= (E^{\epsilon}u, v), \qquad u, v \in \mathscr{D}_{\infty}.$$

The first two equalities in (5.21) follow from (5.1) and the fact that the function $G^{(\nu)}(t, \tau, \lambda)u(\tau)\overline{v(t)}$ is less than an integrable function for $u, v \in D_{\infty}$. (By Lemma 1

$$|G^{(\nu)}(t, \tau, \lambda)u(\tau)\overline{v(t)}| < \gamma^{\nu}(Kn^2)^{\nu+1}\Phi(t)\Phi(\tau)|u(\tau)||\overline{v(t)}|$$

and $\Phi(t)\Phi(\tau)|u(\tau)||v(t)|$ is integrable when $u, v \in D_{\infty}$.) The third equality in (5.22) is by Theorem 2 and the fourth equality in (5.21) is by (2.5). From (5.6) and (5.22) it follows that $E^{\epsilon}(\Delta)$ is a bounded analytic operator by a theorem of Frechet (**6**, p. 385).

**6. Weakened assumptions.** The restrictions placed on $q$ in preceding sections may be weakened. In fact Theorems 3 and 4 remain valid when assumption (i) is replaced by

$$\text{(i)}'\qquad \int_0^{\infty} \Phi_1^2(t)|q(t)|^{\nu}dt < \gamma_1 < \infty \qquad \nu = 1, 2, \ldots,$$

where $\Phi_1(t) = \sup |s_j(t, l)|, j = 1, \ldots, n, l \in \Delta$. It is not necessary to assume $q$ bounded. We shall omit giving the details of the proof of how Theorems 3 and 4 follow from (i)′ but simply outline the necessary steps in the argument: First of all one observes, by reviewing the proof of Theorems 3 and 4, that the series $\hat{E}^{\epsilon}(\Delta) = \sum \epsilon^{\nu}E^{\nu}(\Delta)$ represents a bounded operator for $|\epsilon| < (\gamma_1 K n^2)^{-1}$. It remains to redefine $H_{\epsilon}$, show it self adjoint with domain $\mathscr{D}_{H_0}$, and show that $\hat{E}^{\epsilon}(\Delta)$ is the spectral measure of $H_{\epsilon}$. To define $H_{\epsilon}$ one shows that $\mathscr{G}^{\epsilon}(\lambda)$, defined in Theorem 2, is a bounded operator for $|\epsilon| < (\gamma_1 K n^2)^{-1}$, $\mathscr{I}(\lambda) > 4$ using (i)′. Then $H_{\epsilon} - \lambda 1$ is defined to be the inverse of $\mathscr{G}^{\epsilon}(\lambda)$. Using properties of $\mathscr{G}^{\epsilon}(\lambda)$ one shows $H_{\epsilon}$ is self adjoint,

$$\mathscr{D}_{H_{\epsilon}} = \mathscr{D}_{H_0}, \quad L_{\epsilon}u = H_{\epsilon}u, \quad u \in \mathscr{D}_{H_0}.$$

Finally to show that $\hat{E}^{\epsilon}(\Delta)$ is the spectral measure corresponding to $H_{\epsilon}$ we use a limiting argument. Define operators $L_{\epsilon}(a, b) = L_0 + \epsilon q(a, b, t)$ where

$$(6.1)\qquad q(a, b, t) = \begin{cases} q(t)\Phi_1^2(t)/\Phi_b^2(t), & t < a \\ 0, & t > a \end{cases}$$

and $\Phi_b(t) = \sup |s_j(t, \lambda)|$, $j = 1, \ldots, n$, $l \in \Delta$, $0 < \delta < b$. The operators $L_\epsilon(a, b)$ satisfy assumption (i) so that Theorems 2, 3, and 4 hold for $L_\epsilon(a, b)$, $|\epsilon| < (\gamma_1 K n^2)^{-1}$. Now the resolvent $\mathcal{G}^\epsilon(\lambda, a, b)$ of $H_\epsilon(a, b)$ converges strongly to the resolvent $\mathcal{G}^\epsilon(\lambda)$ of $H_\epsilon$, $a \to \infty$, $b \to 0$. By a well-known theorem of Rellich the spectral measure $E^\epsilon(\Delta, a, b)$ converges strongly to $E^\epsilon(\Delta)$, $a \to \infty$, $b \to 0$. On the other hand, $E^\epsilon(\Delta, a, b)$ converges strongly to $\hat{E}^\epsilon(\Delta)$ so $E^\epsilon(\Delta) = \hat{E}^\epsilon(\Delta)$.

Note that the results of § 5 hold if $L_0$ has a singular point at $t = 0$ since the boundary conditions there are given in the abstract form **(6)**.

It is important to consider weakening assumption (iii). An alternative assumption is the following:

(iii)′ There exists a unimodular matrix $V_j^k(\lambda)$ which is analytic in $\lambda$, $l \in \Delta$, $-\delta_0 < \delta < \delta_0$, $\bar{V}_j^k(\lambda) = V_j^k(\bar{\lambda})$ such that the spectral density matrix $\tilde{\rho}^{jk}(l)$ defined by

$$\tilde{\rho}^{jk}(l) = \int_0^l V_r^j(l) V_s^k(l) d\rho^{rs}(l) \qquad \Delta = [\alpha, \beta] \tag{6.2}$$

is a diagonal matrix. We may derive Theorem 3 using (iii)′ in place of (iii) simply by using $\tilde{\rho}^{jk}$ in place of $\rho^{jk}$ and also $\tilde{s}_j = U_j^k s_k$, $\tilde{M}^{jk} = V_r^j V_s^k M^{rs}$ in place of $s_j$, $M^{jk}$. ($U_j^k$ means the inverse of $V_k^j$ **(5**, p. 536**)**.

An alternative to assumption (iv) is the following set of three conditions:
(iv)′ $M^{jk} = 0, j = r + 1, \ldots, n$.
(iv)″ if $s_{j+p}'$ are permutations of the regular solutions $s_j$, $j = 1, \ldots, r$ according to the rules $s_{j+p}' = s_{j+p}$, $j + p \leqslant r$, $s_{j+p}' = s_{j+p-r}$, $j + p > r$ then for $p = 1, \ldots, r$

$$\int_\Delta s'_{j+p}(t, l) s'_{k+p}(\tau, l) d\rho^{jk}(l)$$

are kernels of bounded operators with bound $P^2$.

(iv)‴ for $k = r + 1, \ldots, n$.

$$\int_0^\infty \left( |M^{jk}| |s_j|^2 \left( \int_0^t |s_k|^2 dt \right) \right) |q| dt < P^2, \qquad l \in \Delta.$$

We may derive Theorem 3 with (iv)′, (iv)″, and (iv)‴ in place of assumption (iv) by minor modifications of the argument. Formulas (5.19) and (5.20) must be re-proved using (iv)″ when $i = 1$ and (iv)‴ when $i = 2$.

For the case $n = 4$ and $L_0 = d^4/dt^4$, $\psi_{01} = t$, $\psi_{02} = t^3/3!$ assumptions (ii), (iii)′, (iv)′, (iv)″, and (iv)‴ are satisfied with $r = 3$ provided $\Delta = [\alpha, \beta]$ is any interval of the form $0 < \alpha \leqslant l \leqslant \beta < \infty$. The expansion theorem for this case has been obtained by Windau **(10)**. Using Windau's results one may easily verify that assumptions (ii), (iii)′, (iv)′, (iv)″, (iv)‴ hold.

## References


1. F. H. Brownell, *Spectrum of the static potential Schrödinger equation over $E_n$*, Ann. Math., *54* (1951), 554–594.
2. ——— *Perturbation of the n-dimensional Schrödinger differential operator*, Abstract no. 430t, Bull. Amer. Math. Soc., *60* (1954). 346.
3. E. A. Coddington, *On self-adjoint ordinary differential operators*, Math. Scand., *4* (1956), 9–21.
4. ——— *The spectral matrix and Green's functions for singular self-adjoint boundary value problems*, Can. J. Math., *6* (1954), 169–185.
5. K. Kodaira, *On ordinary differential equations of any even order and the corresponding eigenfunction expansions*, Amer. J. Math., *72* (3) (1950), 502–543.
6. J. Moser, *Störungstheorie des kontinuierlichen Spektrums*, Math. Ann. *125* (1953), 366–393.
7. M. Rosenblum, *Perturbation of the continuous spectrum and unitary equivalence*, Pac. J. Math., 7 (1957), 997–1010.
8. M. H. Stone, *Linear transformations in Hilbert space*, Amer. Math. Soc. Coll. Pub., *15* (1932).
9. E. C. Titchmarsh, *Eigenfunction expansions, Part II*, (London: Oxford University Press, 1958).
10. W. Windau, *Ueber lineare differentialgleichungen vierter ordnung mit singularitäten und die dazugehörigen darstellungen willkürlicher functionen*, Math. Ann., *83* (1921), 256–279.


*University of Arizona*

# SUR LES REPRESENTATIONS UNITAIRES DES GROUPES DE LIE NILPOTENTS. VI

JACQUES DIXMIER

Soient $\Gamma$ un groupe localement compact, $\hat{\Gamma}$ l'ensemble des classes de représentations unitaires irréductibles de $\Gamma$. Dans **(4)**, Godement a défini une topologie sur $\hat{\Gamma}$ que nous appellerons "topologie canonique." Cette topologie est facile à étudier lorsque $\Gamma$ est abélien ou compact, mais fort mal connue en général. Dans un article récent **(3)**, Fell a repris l'étude de la topologie canonique de $\hat{\Gamma}$, et réussi à la calculer lorsque $\Gamma$ est le groupe spécial linéaire complexe à $n$ variables. L'espace $\hat{\Gamma}$ est alors "presque" séparé, et, par suite, "presque" localement compact.

Il semble nécessaire d'étudier complètement quelques cas particuliers avant d'entreprendre une théorie générale de $\hat{\Gamma}$. Dans un article antérieur **(1)**, j'ai déterminé *l'ensemble* $\hat{\Gamma}$ lorsque $\Gamma$ est un groupe de Lie nilpotent simplement connexe de dimension $\leqslant 5$ (il y a essentiellement 8 tels groupes notés $\Gamma_3$, $\Gamma_4$, $\Gamma_{5,1}$, $\Gamma_{5,2}, \ldots, \Gamma_{5,6}$ dans **(1)**; nous conserverons ces notations ici). Nous allons calculer les topologies canoniques de $\hat{\Gamma}_3$, $\hat{\Gamma}_4$, $\hat{\Gamma}_{5,1}$, $\hat{\Gamma}_{5,2}$, $\hat{\Gamma}_{5,3}$, et $\hat{\Gamma}_{5,6}$, et obtenir un résultat partiel pour $\hat{\Gamma}_{5,5}$ (le cas de $\Gamma_{5,4}$ conduit à des calculs qui m'ont rebuté). Bien que l'écart entre les topologies trouvées et la séparation soit plus sévère que dans le cas de Fell, les espaces obtenus sont encore très raisonnables. La méthode des éléments "boundedly represented" de Fell ne semble pas utilisable dans notre cas; mais il nous suffira d'appliquer le premier chapitre de **(3)**, en le combinant avec quelques calculs explicites, assez variables d'un cas à l'autre.

Les résultats conduisent à un certain nombre de conjectures. Par exemple:

(1) Soient $\Gamma$ un groupe de Lie nilpotent simplement connexe, $\mathfrak{g}$ son algèbre de Lie, $\mathfrak{U}(\mathfrak{g})$ l'algèbre enveloppante de $\mathfrak{g}$, $\mathfrak{Z}(\mathfrak{g})$ le centre de $\mathfrak{U}(\mathfrak{g})$. Tout $a \in \mathfrak{Z}(\mathfrak{g})$ définit une fonction scalaire $f_a$ sur $\hat{\Gamma}$. Alors $f_a$ est continue. (Ceci est maintenant démontré; cf. P. Bernat et J. Dixmier, C. R. Acad. Sci. Paris, 1960).

(2) Soit $P$ l'ensemble des points $U$ de $\hat{\Gamma}$ possédant la propriété suivante: pour tout $V \in \hat{\Gamma}$, $V \neq U$, il existe des voisinages de $U$ et $V$ qui sont disjoints. Alors, $P$ est une partie ouverte partout dense de $\hat{\Gamma}$, et la topologie induite sur $P$ par la topologie canonique de $\hat{\Gamma}$ est la moins fine rendant continues les fonctions $f_a$.

D'après **(2)**, les caractères des groupes nilpotents simplement connexes sont des distributions tempérées. Nous les calculons pour $\Gamma_4$ et $\Gamma_{5,5}$, ce qui fait apparaître des fonctions de Bessel.

Notations. On désignera par **R** le corps des nombres réels, par **C** le corps

Reçu le 17 mars, 1959.

des nombres complexes, par $L_{\mathbf{C}}^2(\mathbf{R}^n)$ l'ensemble des fonctions complexes sur $\mathbf{R}^n$ de carré intégrable pour la mesure de Lebesgue, par $\mathscr{S}(\mathbf{R}^n)$ l'ensemble des fonctions complexes sur $\mathbf{R}^n$ indéfiniment différentiables à décroissance rapide. Si $\Gamma$ est un groupe de Lie nilpotent simplement connexe, il y a une notion intrinsèque de fonction polynôme sur $\Gamma$, donc $\mathscr{S}(\Gamma)$ a encore un sens. On désignera par $\mathscr{F}$ la transformation de Fourier. Si $F \in \mathscr{S}(\mathbf{R}^4)$, on notera $\hat{F}_{24}$ la fonction

$$(\xi_1, \xi_2, \xi_3, \xi_4) \rightarrow \iint F(\xi_1, \eta_2, \xi_3, \eta_4) \exp - i(\xi_2\eta_2 + \xi_4\eta_4)d\eta_2 d\eta_4,$$

transformée de Fourier de $F$ par rapport aux 2e et 4e variables; et on définira de manière analogue, pour $F \in \mathscr{S}(\mathbf{R}^n)$, la fonction

$$\hat{F}_{i_1 i_2 \ldots i_p} \quad \text{pour} \quad 1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n.$$

Si $\beta_1, \beta_2, \ldots, \beta_n \in \mathbf{R}$, on notera $\mathscr{T}(\beta_1, \ldots, \beta_n)$ l'opérateur unitaire dans $L_{\mathbf{C}}^2(\mathbf{R}^n)$ défini par

$$(\mathscr{T}(\beta_1, \ldots, \beta_n)f) \quad (\xi_1, \ldots, \xi_n) = f(\xi_1 + \beta_1, \ldots, \xi_n + \beta_n).$$

Si $h$ est une fonction complexe mesurable essentiellement bornée sur $\mathbf{R}^n$, on notera $\mathscr{M}(h)$ (ou $\mathscr{M}(h(\xi_1, \ldots, \xi_n))$ par abus de notation) l'opérateur continu dans $L_{\mathbf{C}}^2(\mathbf{R}^n)$ défini par

$$(\mathscr{M}(h)f)(\xi_1, \ldots, \xi_n) = h(\xi_1, \ldots, \xi_n)f(\xi_1, \ldots, \xi_n).$$

## 1. Quelques lemmes.

LEMME 1. *Soient $A$ une $C^*$-algèbre, $B$ une partie partout dense de $A$; soient $T_0$ un point du spectre $\hat{A}$ de $A$, et $S \subset \hat{A}$. On suppose que, pour tout $x \in B$, on a $\sup_{T \in S} \|T(x)\| \geqslant \|T_0(x)\|$. Alors $T_0$ est canoniquement adhérent à $S$.*

*Démonstration.* L'inégalité $\sup_{T \in S}\|T(x)\| \geqslant \|T_0(x)\|$ s'étend par continuité à tout $x \in A$. Le lemme résulte alors du lemme 2.1 de **(3)**.

LEMME 2. *Soient $A$ une $C^*$-algèbre, $B$ une partie partout dense de $A$, et $\mathscr{T}$ une topologie sur $\hat{A}$. On suppose que, pour tout $x \in B$, la fonction $T \rightarrow \|T(x)\|$ est continue pour $\mathscr{T}$. Alors $\mathscr{T}$ est plus fine que la topologie canonique de $A$.*

*Démonstration.* Soit $S$ une partie canoniquement fermée de $\hat{A}$. Si $T_0 \in \hat{A}$ est adhérent à $S$ pour $\mathscr{T}$, on a $\sup_{T \in S}\|T(x)\| \geqslant \|T_0(x)\|$ pour tout $x \in B$. Donc (Lemme 1) $T_0$ est canoniquement adhérent à $S$ et par suite $T_0 \in S$. Donc $S$ est fermée pour $\mathscr{T}$.

LEMME 3. *Soient $\Gamma$ un groupe localement compact, $\Gamma'$ un sous-groupe distingué fermé de $\Gamma$, et $\varphi$ l'application canonique de $\Gamma$ sur $\Gamma/\Gamma'$. Alors l'application $\hat{\varphi}: U \rightarrow U \circ \varphi$ de $(\Gamma/\Gamma')^-$ dans $\hat{\Gamma}$ est un homéomorphisme de $(\Gamma/\Gamma')^-$ sur une partie fermée de $\hat{\Gamma}$.*

*Démonstration.* Le fait que $\hat{\varphi}$ soit un homéomorphisme de $(\Gamma/\Gamma')^-$ sur une partie $S$ de $\hat{\Gamma}$ résulte aussitôt du Théorème 1.5 de **(3)** et du fait qu'une partie

de $\Gamma/\Gamma'$ est compacte si et seulement si elle est l'image par $\varphi$ d'une partie compacte de $\Gamma$. D'autre part, si $T \in \hat{\Gamma}$ est canoniquement adhérent à $S$, le Théorème 1.5 de **(3)** montre que les fonctions de type positif associées à $T$ sont constantes sur les classes modulo $\Gamma'$. En particulier, si $x$ est un vecteur unitaire de l'espace où opère $T$, on a $(T(\gamma)x|x) = 1$ pour $\gamma \in \Gamma'$, donc $T(\gamma)x = x$ pour $\gamma \in \Gamma'$, donc $T(\gamma) = 1$ pour $\gamma \in \Gamma'$. Donc $T \in S$.

LEMME 4. *Soient $F \in \mathscr{S}(\mathbf{R}^n)$, $\Lambda$ un espace topologique, $\lambda_0 \in \Lambda$, et $\varphi$ une application continue de $\mathbf{R}^p \times \Lambda$ dans $\mathbf{R}^n$ possédant la propriété suivante: il existe un $K > 0$, un $\alpha > 0$, et un voisinage $V$ de $\lambda_0$ dans $\Lambda$ tel que $||\varphi(x, \lambda)|| \geqslant K||x||^\alpha$ pour $||x|| \geqslant 1$ et $\lambda \in V$. Posons $\psi_\lambda(x) = F(\varphi(x, \lambda))$. Alors, quand $\lambda \to \lambda_0$,*

$$\psi_\lambda \to \psi_{\lambda_0}$$

*dans $L_{\mathbf{C}}^2(\mathbf{R}^p)$.*

*Démonstration.* Soit $q = (p + 1/)2\alpha$. Il existe une constante $k > 0$ telle que $|F(y)|\ (1 + ||y||^q) < k$. Alors, pour $\lambda \in V$ et $x \in \mathbf{R}^p$, $||x|| \geqslant 1$, on a

$$|\psi_\lambda(x)| = |F(\varphi(x, \lambda))| < \frac{k}{1 + ||\varphi(x, \lambda)||^q} < \frac{k}{1 + K^q||x||^{\alpha q}} = \frac{k}{1 + K^q||x||^{\frac{1}{2}(p+1)}}.$$

Donc $\psi_\lambda \in L_{\mathbf{C}}^2(\mathbf{R}^p)$, et

$$|\psi_\lambda - \psi_{\lambda_0}|$$

est, pour $\lambda \in V$, majoré par une fonction fixe de $L_{\mathbf{C}}^2(\mathbf{R}^p)$. Comme

$$\psi_\lambda(x) - \psi_{\lambda_0}(x) \to 0$$

pour chaque valeur de $x$ quand $\lambda \to \lambda_0$, on voit que

$$\int |\psi_\lambda(x) - \psi_{\lambda_0}(x)|^2 dx \to 0$$

quand $\lambda \to \lambda_0$.

**2. Topologie de $\hat{\Gamma}_3$.** D'après **(1)**, proposition 3, $\hat{\Gamma}_3$ est réunion de 2 sous-ensembles disjoints $A$ et $B$:

(1) $A$ est l'ensemble des représentations $U_\lambda (\lambda \in \mathbf{R}, \lambda \neq 0)$; chaque représentation $U_\lambda$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$ et est définie par la formule

$$(U_\lambda(\gamma)f)(\theta) = [\exp i\lambda(\rho_3 - \rho_2\theta)]f(\theta + \rho_1)$$
$$(\gamma = (\rho_1, \rho_2, \rho_3) \in \Gamma_3, \quad f \in L_{\mathbf{C}}^2(\mathbf{R}), \quad \theta \in \mathbf{R}).$$

Toutes les fois que cela sera commode, nous identifierons $A$ et $\mathbf{R} - \{0\}$ par l'application $\lambda \to U_\lambda$.

(2) $B$ est l'ensemble des représentations $V_{\lambda,\mu}(\lambda, \mu \in \mathbf{R})$; chaque représentation $V_{\lambda,\mu}$ opère dans un espace de dimension 1, donc s'identifie à une fonction scalaire sur $\Gamma_3$, conformément à la relation

$$V_{\lambda,\mu}(\gamma) = \exp i(\lambda\rho_1 + \mu\rho_2)$$

$(\gamma = (\rho_1, \rho_2, \rho_3) \in \Gamma_3)$. Nous identifierons éventuellement $B$ et $\mathbf{R}^2$ par l'application $(\lambda, \mu) \to V_{\lambda,\mu}$.

Munissons $A$ de la topologie de $\mathbf{R} - \{0\}$ et $B$ de la topologie de $\mathbf{R}^2$. La topologie sur $\hat{\Gamma}_3$ qui fait de $\hat{\Gamma}_3$ la somme topologique des espaces topologiques $A$ et $B$ précédents sera appelée la *topologie des paramètres*. D'après le Lemme 3, $B$ est canoniquement fermé dans $\hat{\Gamma}_3$, et la topologie induite sur $B$ par la topologie canonique de $\hat{\Gamma}_3$ est la topologie des paramètres.

Soit $F \in \mathscr{S}(\Gamma_3)$. Soient $f$ et $g$ des fonctions numériques complexes d'une variable réelle, continues à support compact. On a

$$(1) \quad (U_\lambda(F)f|g)$$
$$= \iiint F(\rho_1, \rho_2, \rho_3)d\rho_1 d\rho_2 d\rho_3 \int \exp i\lambda(\rho_3 - \rho_2\theta)f(\theta + \rho_1)\overline{g(\theta)}\, d\theta$$
$$= \iiiint F(\rho_1 - \theta, \rho_2, \rho_3) \exp i\lambda(\rho_3 - \rho_2\theta)f(\rho_1)\overline{g(\theta)}d\rho_1 d\rho_2 d\rho_3\, d\theta.$$

Donc $U_\lambda(F)$ est défini par le noyau

$$(2) \qquad K_\lambda(\rho_1, \theta) = \iint F(\rho_1 - \theta, \rho_2, \rho_3) \exp i\lambda(\rho_3 - \rho_2\theta)d\rho_2 d\rho_3$$
$$= \hat{F}_{23}(\rho_1 - \theta, \lambda\theta, -\lambda).$$

Lemme 5. *Soit $F \in \mathscr{S}(\Gamma_3)$. La fonction $T \to \|T(F)\|$ est continue sur $\hat{\Gamma}_3$ pour la topologie des paramètres.*

*Démonstration.* La continuité est immédiate sur $B$. D'autre part, en utilisant les notations précédentes, l'application $\lambda \to K_\lambda$ est une application fortement continue de $\mathbf{R} - \{0\}$ dans $L_{\mathbf{C}}^2(\mathbf{R}^2)$ (Lemme 4). Donc l'application $\lambda \to U_\lambda(F)$ est continue pour la norme d'Hilbert-Schmidt des opérateurs et *a fortiori* pour la topologie uniforme. Ceci prouve la continuité de l'application $\lambda \to \|U_\lambda(F)\|$ sur $A$. D'où le lemme.

Utilisant le Lemme 2, on voit que les ensembles canoniquement fermés de $\hat{\Gamma}_3$ sont à chercher parmi les ensembles $A_1 \cup B_1$, où $A_1 \subset A$ et $B_1 \subset B$ sont fermés pour la topologie des paramètres. En outre, $A_1$ et $B_1$ doivent vérifier des conditions supplémentaires:

Lemme 6. *Soient $A_1 \subset A$, $B_1 \subset B$. On suppose $A_1 \cup B_1$ canoniquement fermé. Alors, si 0 est adhérent à $A_1$ dans $\mathbf{R}$, on a $B_1 = B$.*

*Démonstration.* Soit $F \in \mathscr{S}(\Gamma_3)$. Soient $f$ et $g$ des fonctions numériques complexes d'une variable réelle, continues à support compact. Pour tout $\beta \in \mathbf{R}$, on a, d'après (1)

$$\left(U_\lambda(F).\mathscr{T}\left(\frac{\beta}{\lambda}\right)f \middle| \mathscr{T}\left(\frac{\beta}{\lambda}\right)g\right)$$
$$= \iiiint F(\rho_1, \rho_2, \rho_3) \exp i\lambda(\rho_3 - \rho_2\theta)f\left(\theta + \rho_1 + \frac{\beta}{\lambda}\right)\overline{g\left(\theta + \frac{\beta}{\lambda}\right)}\, d\rho_1 d\rho_2 d\rho_3 d\theta$$
$$= \iiiint F(\rho_1, \rho_2, \rho_3) \exp i(\lambda\rho_3 - \lambda\rho_2\theta + \beta\rho_2)f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 d\rho_2 d\rho_3 d\theta$$

Quand $\lambda \to 0$, ceci tend vers

$$\iiiint F(\rho_1, \rho_2, \rho_3) \exp(i\,\beta\rho_2) f(\theta + \rho_1)\overline{g(\theta)} d\rho_1 d\rho_2 d\rho_3 d\theta$$
$$= \iint \hat{F}_{22}(\rho_1, -\beta, 0) f(\theta + \rho_1)\overline{g(\theta)} d\rho_1 d\theta = (Vf|g),$$

où $V$ est, dans $L_{\mathbf{C}}^2(\mathbf{R})$, l'opérateur de convolution avec la fonction $\rho_1 \to \hat{F}_{23}(-\rho_1, -\beta, 0)$. Ainsi, quand $\lambda \to 0$,

$$\mathscr{T}\left(\frac{\beta}{\lambda}\right)^{-1} U_\lambda(F) \mathscr{T}\left(\frac{\beta}{\lambda}\right)$$

tend faiblement vers $V$, donc $\underline{\lim}_{\lambda\to 0}||U_\lambda(F)|| \geqslant ||V||$. Par transformation de Fourier, on voit que

$$||V|| = \sup_{\xi\in\mathbf{R}} |\hat{F}_{123}(\xi, -\beta, 0)| = \sup_{\xi\in\mathbf{R}} ||V_{-\xi,\beta}(F)||.$$

Donc $\underline{\lim}_{\lambda\to 0}||U_\lambda(F)|| \geqslant ||V_{\xi,\mu}(F)||$ quels que soient $\xi \in \mathbf{R}$, $\mu \in \mathbf{R}$. Le Lemme 6 résulte alors du Lemme 1.

PROPOSITION 1. *Les ensembles canoniquement fermés dans* $\Gamma_3$ *sont les ensembles* $A_1 \cup B_1$ $(A_1 \subset A, B_1 \subset B)$ *possédant les propriétés suivantes:*

(1) $A_1$ *est fermé dans* $A$ *pour la topologie des paramètres;*

(2) $B_1$ *est fermé dans* $B$ *pour la topologie des paramètres;*

(3) *Si* 0 *est adhérent à* $A_1$ *dans* $\mathbf{R}$, *on a* $B_1 = B$.

*Démonstration*. Nous savons déjà que les conditions (1), (2), et (3) sont nécessaires pour que $A_1 \cup B_1$ soit canoniquement fermé. Supposons ces conditions remplies et montrons que $A_1 \cup B_1$ est canoniquement fermé. Soit $T \in \Gamma_3$, $T \notin A_1 \cup B_1$. Il s'agit de montrer que $T$ n'est pas canoniquement adhérent à $A_1 \cup B_1$. Or $T$ n'est pas canoniquement adhérent à $B_1$ car $B_1$ est canoniquement fermé. Il s'agit donc de prouver que $T$ n'est pas canoniquement adhérent à $A_1$. Nous allons pour cela construire une $F \in \mathscr{S}(\Gamma_3)$ telle que $T(F) \neq 0$, et telle que $T'(F) = 0$ pour $T' \in A_1$. Distinguons deux cas.

(A) $$T = U_{\lambda_0} \notin A_1.$$

Soient $\theta \in \mathbf{R}$, $\rho_1 \in \mathbf{R}$. Soit $E$ l'ensemble des points de $\mathbf{R}^3$ dont la 3e coordonnée appartient à $-A_1$. Puisque $A_1$ est fermé pour la topologie des paramètres, le point $(\rho_1 - \theta, \lambda_0\theta, -\lambda_0)$ est extérieur à $E$. Soit $G$ une fonction de $\mathscr{S}(\mathbf{R}^3)$ non nulle en ce point et nulle sur $E$. On a $G = \hat{F}_{23}$ pour $F \in \mathscr{S}(\mathbf{R}^3)$ bien choisie. Alors, compte tenu de (2), $U_{\lambda_0}(F)$ est défini par un noyau de $\mathscr{S}(\mathbf{R}^2)$ qui n'est pas identiquement nul, donc

$$U_{\lambda_0}(F) \neq 0;$$

par contre, pour $\lambda \in A_1$, $U_\lambda(F)$ est défini par le noyau 0, donc $U_\lambda(F) = 0$.

(B) $$T = V_{\lambda_0,\mu_0} \notin B_1.$$

D'après l'hypothèse (3) de la proposition, 0 n'est pas adhérent à $A_1$ dans

**R**. Le point $(-\lambda_0, -\mu_0, 0)$ est donc extérieur à l'ensemble $E$ introduit plus haut. Il existe une $F \in \mathscr{S}(\mathbf{R}^3)$ telle que $U_\lambda(F) = 0$ pour $\lambda \in A_1$, et $\hat{F}_{123}(-\lambda_0, -\mu_0, 0) \neq 0$, donc

$$V_{\lambda_0,\mu_0}(F) \neq 0.$$

*Remarque.* Le quotient de $\Gamma_3$ par un sous-groupe discret non trivial de son centre a donc un dual localement compact, qui s'identifie à la somme topologique de $\mathbf{R}^2$ et de $\mathbf{Z} - \{0\}$ ($\mathbf{Z}$: ensemble des entiers rationnels).

**3. Topologie de $\hat{\Gamma}_4$.** D'après **(1)**, Proposition 4, $\hat{\Gamma}_4$ est réunion de 3 sous-ensembles disjoints $A$, $B$, $C$:

(1) $A$ est l'ensemble des $U_{\lambda,\mu}$ ($\lambda, \mu \in \mathbf{R}, \lambda \neq 0$); chaque $U_{\lambda,\mu}$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$ et est définie par la formule

$$(U_{\lambda,\mu}(\gamma)f)(\theta) = \exp i\left(-\frac{1}{2}\frac{\mu}{\lambda}\rho_2 + \lambda\rho_4 - \lambda\rho_3\theta + \tfrac{1}{2}\lambda\rho_2\theta^2\right)f(\theta + \rho_1)$$

$$(\gamma = (\rho_1, \rho_2, \rho_3, \rho_4) \in \Gamma_4, f \in L^2_{\mathbf{C}}(\mathbf{R}), \theta \in \mathbf{R}).$$

(2) $B$ est l'ensemble des $V_\lambda$ ($\lambda \in \mathbf{R}, \lambda \neq 0$); chaque $V_\lambda$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$ et est définie par la formule

$$(V_\lambda(\gamma)f)(\theta) = \exp i\lambda(\rho_3 - \rho_2\theta)f(\theta + \rho_1).$$

(3) $C$ est l'ensemble des $W_{\lambda,\mu}$ ($\lambda, \mu \in \mathbf{R}$); chaque $W_{\lambda,\mu}$ s'identifie à une fonction scalaire sur $\Gamma_4$ conformément à la relation

$$W_{\lambda,\mu}(\gamma) = \exp i(\lambda\rho_1 + \mu\rho_2).$$

D'après le Lemme 3, $B \cup C$ est canoniquement fermé dans $\hat{\Gamma}_4$, et la topologie induite sur $B \cup C$ par la topologie canonique de $\hat{\Gamma}_4$ s'identifie à la topologie canonique de $\hat{\Gamma}_3$; elle est donc connue par la Proposition 1.

On définit sur $\hat{\Gamma}_4$ une "topologie des paramètres" comme on l'a fait sur $\hat{\Gamma}_3$.

Nous noterons $\Pi_+$ (resp. $\Pi_-$) l'ensemble des $(\lambda, \mu) \in \mathbf{R}^2$ tels que $\lambda > 0$ (resp. $\lambda < 0$). Nous identifierons $A$ à $\Pi_+ \cup \Pi_-$, $B$ à $\mathbf{R} - \{0\}$ et $C$ à $\mathbf{R}^2$ quand cela sera commode.

Soit $F \in \mathscr{S}(\Gamma_4)$. Soient $f$ et $g$ deux fonctions numériques complexes d'une variable réelle, continues à support compact. On a

$$(3)\quad (U_{\lambda,\mu}(F)f|g)$$

$$= \int \cdots \int F(\rho_1, \ldots, \rho_4) \exp i\left(-\frac{1}{2}\frac{\mu}{\lambda}\rho_2 + \lambda\rho_4 - \lambda\rho_3\theta + \tfrac{1}{2}\lambda\rho_2\theta^2\right)f(\theta + \rho_1)\overline{g(\theta)}\, d\rho_1 \ldots d\rho_4 d\theta$$

$$= \int \cdots \int F(\rho_1 - \theta, \rho_2, \rho_3, \rho_4) \exp i\left(-\frac{1}{2}\frac{\mu}{\lambda}\rho_2 + \lambda\rho_4 - \lambda\rho_3\theta + \tfrac{1}{2}\lambda\rho_2\theta^2\right) f(\rho_1)\overline{g(\theta)}\,d\rho_1 \ldots d\rho_4 d\theta.$$

Donc $U_{\lambda,\mu}(F)$ est défini par le noyau

(4) $K_{\lambda,\mu}(\rho_1, \theta)$

$$= \iiint F(\rho_1 - \theta, \rho_2, \rho_3, \rho_4) \exp i\left(-\frac{1}{2}\frac{\mu}{\lambda}\rho_2 + \lambda\rho_4 - \lambda\rho_3\theta + \tfrac{1}{2}\lambda\rho_2\theta^2\right)d\rho_2 d\rho_3 d\rho_4$$

$$= \hat{F}_{234}\left(\rho_1 - \theta, \frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2, \lambda\theta, -\lambda\right).$$

LEMME 7. *Soit $F \in \mathscr{S}(\Gamma_4)$. La fonction $T \to ||T(F)||$ est continue sur $\Gamma_4$ pour la topologie des paramètres.*

*Démonstration.* La continuité sur $B \cup C$ résulte du Lemme 5. D'autre part, en utilisant les notations précédentes, l'application $(\lambda, \mu) \to K_{\lambda,\mu}$ est une application fortement continue de $A$ dans $L_{\mathbf{C}}^2(\mathbf{R}^2)$ (Lemme 4). On en conclut comme au Lemme 5 que l'application $(\lambda, \mu) \to ||U_{\lambda,\mu}(F)||$ est continue sur $A$.

LEMME 8. *Soient $A_1 \subset A$, $B_1 \subset B$, $C_1 \subset C$. On suppose $A_1 \cup B_1 \cup C_1$ canoniquement fermé. Alors:*

(1) *Si $\mu_0 > 0$ est tel que $(0, \mu_0)$ soit adhérent à $A_1$ dans $\mathbf{R}^2$, on a*

$$V\mu_0^{\frac{1}{2}} \in B_1, \; V - \mu_0^{\frac{1}{2}} \in B_1.$$

(2) *Si $(0, 0)$ est adhérent à $A_1 \cap \Pi^+$ dans $\mathbf{R}^2$, on a $W_{\lambda_0,\mu_0} \in C_1$ pour*

$$\mu_0 > \underline{\lim}_{(\lambda,\mu)\to(0,0),(\lambda,\mu)\in A_1\cap\Pi_+}\left(-\frac{\mu}{2\lambda}\right).$$

(3) *Si $(0, 0)$ est adhérent à $A_1 \cap \Pi_-$ dans $\mathbf{R}^2$, on a $W_{\lambda_0,\mu_0} \in C_1$ pour*

$$\mu_0 < \overline{\lim}_{(\lambda,\mu)\to(0,0),(\lambda,\mu)\in A_1\cap\Pi_-}\left(-\frac{1}{2}\frac{\mu}{\lambda}\right).$$

*Démonstration.* Soit $F \in \mathscr{S}(\Gamma_4)$. Soient $f$ et $g$ des fonctions numériques complexes d'une variable réelle, continues à support compact. Pour $\mu > 0$, on a, d'après (3)

$$\left(U_{\lambda,\mu}(F)\mathscr{T}\left(\frac{\mu^{\frac{1}{2}}}{\lambda}\right)f \,\middle|\, \mathscr{T}\left(\frac{\mu^{\frac{1}{2}}}{\lambda}\right)g\right) = \int \ldots \int F(\rho_1, \ldots, \rho_4)\exp(i\Delta)f(\theta + \rho_1) \overline{g(\theta)}\, d\rho_1 \ldots d\rho_4\, d\theta$$

avec

$$\Delta = -\frac{1}{2}\frac{\mu}{\lambda}\rho_2 + \lambda\rho_4 - \lambda\rho_3\left(\theta - \frac{\mu^{\frac{1}{2}}}{\lambda}\right) + \tfrac{1}{2}\lambda\rho_2\left(\theta - \frac{\mu^{\frac{1}{2}}}{\lambda}\right)^2$$

$$= \lambda\rho_4 - \lambda\rho_3\theta + \mu^{\frac{1}{2}}\rho_3 + \tfrac{1}{2}\lambda\rho_2\theta^2 - \mu^{\frac{1}{2}}\rho_2\theta.$$

Quand $\lambda \to 0$ et $\mu \to \mu_0 > 0$, l'intégrale tend vers

$$\int \cdots \int F(\rho_1, \ldots, \rho_4) \exp i\, \mu_0^{\frac{1}{2}}(\rho_3 - \rho_2\theta) f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_4 d\theta = (V_{\mu_0^{\frac{1}{2}}}(F)f|g).$$

Donc $\mathscr{T}(\mu^{\frac{1}{2}}/\lambda)^{-1}U_{\lambda,\mu}(F)\mathscr{T}(\mu^{\frac{1}{2}}/\lambda)$ tend faiblement vers $V_{\mu_0^{\frac{1}{2}}}$, et par suite

$$\underline{\lim}_{(\lambda,\mu)\to(0,\mu_0)} ||U_{\lambda,\mu}(F)|| \geqslant ||V_{\mu_0^{\frac{1}{2}}}||.$$

Si $(0, \mu_0)$ est adhérent à $A_1$ dans $\mathbf{R}^2$, on a

$$V_{\mu_0^{\frac{1}{2}}} \in B_1$$

(Lemme 1). Considérant $\mathscr{T}(-\mu^{\frac{1}{2}}/\lambda)^{-1}U_{\lambda,\mu}(F)\mathscr{T}(-\mu^{\frac{1}{2}}/\lambda)$, on voit de même que $V_{-\mu_0^{\frac{1}{2}}} \in B_1$.

Supposons maintenant que $(0,0)$ soit adhérent à $A_1 \cap \Pi_+$ dans $\mathbf{R}^2$. Soit $\mu_0$ un nombre réel fini tel que

$$\mu_0 > \underline{\lim}_{(\lambda,\mu)\to(0,0),\,(\lambda,\mu)\in A_1\cap\Pi_+}\left(-\frac{1}{2}\frac{\mu}{\lambda}\right).$$

Il existe des suites de nombres réels $\lambda_n, \mu_n, \beta_n$ tels que

$$(\lambda_n, \mu_n) \to (0,0),\ (\lambda_n, \mu_n) \in A_1 \cap \Pi_+,\ -\frac{1}{2}\frac{\mu_n}{\lambda_n} + \tfrac{1}{2}\lambda_n\beta_n^2 \to \mu_0.$$

Alors,

$$(U_{\lambda_n,\mu_n}(F)\mathscr{T}(\beta_n)f|\mathscr{T}(\beta_n)g) = \int \cdots \int F(\rho_1, \ldots, \rho_4)\exp(i\Delta)f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_4 d\theta$$

avec

$$\Delta = -\frac{1}{2}\frac{\mu_n}{\lambda_n}\rho_2 + \lambda_n\rho_4 - \lambda_n\rho_3(\theta - \beta_n) + \tfrac{1}{2}\lambda_n\rho_2(\theta - \beta_n)^2$$
$$= \left(-\frac{1}{2}\frac{\mu_n}{\lambda_n} + \tfrac{1}{2}\lambda_n\beta_n^2\right)\rho_2 + \lambda_n\rho_4 - \lambda_n\rho_3\theta + \tfrac{1}{2}\lambda_n\rho_2\theta^2 + \lambda_n\beta_n(\rho_3 - \rho_2\theta).$$

Remarquons que $-\frac{1}{2}\mu_n + \frac{1}{2}\lambda_n^2\beta_n^2 \to 0$, donc que $\lambda_n\beta_n \to 0$, quand $n \to +\infty$. Ceci posé, l'intégrale tend vers

$$\int \cdots \int F(\rho_1, \ldots, \rho_4)\exp(i\mu_0\rho_2)f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_4 d\theta$$
$$= \iint \hat{F}_{234}(\rho_1, -\mu_0, 0, 0)f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 d\theta = (Vf|g)$$

où $V$ est, dans $L_{\mathbf{C}}^2(\mathbf{R})$, l'opérateur de convolution avec la fonction $\rho_1 \to \hat{F}_{234}(-\rho_1, -\mu_0, 0, 0)$. Quand $n \to +\infty$,

$$\mathscr{T}(\beta_n)^{-1}U_{\lambda_n,\mu_n}(F)\mathscr{T}(\beta_n)$$

tend faiblement vers $V$, donc

$$\underline{\lim}_{n\to+\infty}\|U_{\lambda_n,\mu_n}(F)\| \geqslant \sup_{\xi\in\mathbf{R}}|\hat{F}_{1234}(-\xi, -\mu_0, 0, 0)| = \sup_{\xi\in\mathbf{R}}\|W_{\xi,\mu_0}(F)\|.$$

Donc $W_{\xi,\mu_0} \in C_1$ pour tout $\xi \in \mathbf{R}$. On raisonne de façon analogue lorsque $(0, 0)$ est adhérent à $A_1 \cap \Pi_-$ dans $\mathbf{R}^2$.

Le Lemme 8 donne des conditions nécessaires pour que $A_1 \cup B_1 \cup C_1$ soit canoniquement fermé. Pour prouver que ces conditions sont aussi suffisantes, nous aurons besoin du lemme suivant.

Lemme 9. *Soit $A_1$ une partie de $A$ fermée pour la topologie des paramètres. Soit $E$ l'ensemble des points de $\mathbf{R}^4$ de la forme*

$$\left(t, \frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2, \lambda\theta, -\lambda\right),$$

*où $t \in \mathbf{R}$, $\theta \in \mathbf{R}$, $(\lambda, \mu) \in A_1$. Alors l'adhérence de $E$ dans $\mathbf{R}^4$ est contenue dans la réunion des ensembles suivants:*

(1) *$E$;*

(2) *l'ensemble des points $(t, \tau, \pm \mu_0^{\frac{1}{2}}, 0)$, où $t \in \mathbf{R}$, $\tau \in \mathbf{R}$, et où $\mu_0 > 0$ est tel que $(0, \mu_0)$ soit adhérent à $A_1$ dans $\mathbf{R}^2$;*

(3) *si $(0, 0)$ est adhérent à $A_1 \cap \Pi_+$ dans $\mathbf{R}^2$, l'ensemble des points $(t, \tau, 0, 0)$, où*

$$\tau \leqslant \overline{\lim}_{(\lambda,\mu)\to(0,0),\,(\lambda,\mu)\in A_1\cap\Pi_+}\left(\frac{1}{2}\frac{\mu}{\lambda}\right);$$

(4) *si $(0, 0)$ est adhérent à $A_1 \cap \Pi_-$ dans $\mathbf{R}^2$, l'ensemble des points $(t, \tau, 0, 0)$, où*

$$\tau \geqslant \underline{\lim}_{(\lambda,\mu)\to(0,0),\,(\lambda,\mu)\in A_1\cap\Pi_-}\left(\frac{1}{2}\frac{\mu}{\lambda}\right).$$

(En fait, l'énoncé précédent fournit exactement l'adhérence de $E$; mais nous n'aurons pas besoin de ce fait.)

*Démonstration*. Supposons que

$$t_n \to \xi_1, \frac{1}{2}\frac{\mu_n}{\lambda_n} - \tfrac{1}{2}\lambda_n\theta_n^2 \to \xi_2, \lambda_n\theta_n \to \xi_3, -\lambda_n \to \xi_4$$

($t_n \in \mathbf{R}$, $\theta_n \in \mathbf{R}$, $(\lambda_n, \mu_n) \in A_1$), et montrons que $(\xi_1, \xi_2, \xi_3, \xi_4)$ appartient à l'un des ensembles du lemme. Si $\xi_4 \neq 0$, $t_n, \theta_n, \lambda_n, \mu_n$ ont des limites finies $t, \theta, \lambda, \mu$; on a $(\lambda, \mu) \in A_1$ puisque $A_1$ est fermé dans $A$ pour la topologie des paramètres, et

$$\xi_1 = t, \xi_2 = \frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2, \xi_3 = \lambda\theta, \xi_4 = -\lambda,$$

donc $(\xi_1, \xi_2, \xi_3, \xi_4) \in E$. Supposons $\xi_4 = 0$. Comme $\lambda_n^{-1}(\mu_n - \lambda_n^2\theta_n^2) \to 2\xi_2$, on a $\mu_n - \lambda_n^2\theta_n^2 \to 0$, donc $\mu_n \to \xi_3^2$. Si $\xi_3 \neq 0$, alors $\mu_0 = \xi_3^2 > 0$, $(0, \mu_0)$ est

adhérent à $A_1$ dans $\mathbf{R}^2$, et $(\xi_1, \xi_2, \xi_3, \xi_4) = (\xi_1, \xi_2, \pm \mu_0^{\frac{1}{2}}, 0)$. Enfin, si $\xi_3 = \xi_4 = 0$, $(0, 0)$ est adhérent à $A_1$ dans $\mathbf{R}^2$. Supposons $\lambda_n > 0$ pour une infinité de valeurs de $n$ (donc, en changeant de notations, pour toute valeur de $n$). Alors, $(0, 0)$ est adhérent à $A_1 \cap \Pi_+$ dans $\mathbf{R}^2$. On a

$$\xi_2 = \lim\left(\frac{1}{2}\frac{\mu_n}{\lambda_n} - \tfrac{1}{2}\lambda_n\theta_n^2\right) \leqslant \overline{\lim}\left(\frac{1}{2}\frac{\mu_n}{\lambda_n}\right) \leqslant \overline{\lim}_{(\lambda,\mu)\to(0,0),(\lambda,\mu)\in A_1\cap\Pi_+}\left(\frac{1}{2}\frac{\mu}{\lambda}\right).$$

On raisonne de même si $\lambda_n < 0$ pour une infinité de valeurs de $n$.

Proposition 2. *Les ensembles canoniquement fermés dans $\hat{\Gamma}_4$ sont les ensembles $A_1 \cup B_1 \cup C_1$ ($A_1 \subset A$, $B_1 \subset B$, $C_1 \subset C$) possédant les propriétés suivantes:*

(1) *$A_1$ est fermé dans $A$ pour la topologie des paramètres;*

(2) *$B_1 \cup C_1$ est canoniquement fermé dans $B \cup C = \hat{\Gamma}_3$;*

(3) *Si $(0, \mu_0)$ (où $\mu_0 > 0$) est adhérent à $A_1$ dans $\mathbf{R}^2$, on a*

$$V_{\pm\mu_0^{\frac{1}{2}}} \in B_1;$$

(4) *Si $(0, 0)$ est adhérent à $A_1 \cap \Pi_+$ dans $\mathbf{R}^2$, on a*

$$W_{\lambda_0,\mu_0} \in C_1$$

*toutes les fois que*

$$\mu_0 \geqslant \underline{\lim}_{(\lambda,\mu)\to(0,0),(\lambda,\mu)\in A_1\cap\Pi_+}\left(-\frac{1}{2}\frac{\mu}{\lambda}\right);$$

(5) *Si $(0, 0)$ est adhérent à $A_1 \cap \Pi_-$ dans $\mathbf{R}^2$, on a*

$$W_{\lambda_0,\mu_0} \in C_1$$

*toutes les fois que*

$$\mu_0 \leqslant \overline{\lim}_{(\lambda,\mu)\to(0,0),(\lambda,\mu)\in A_1\cap\Pi_-}\left(-\frac{1}{2}\frac{\mu}{\lambda}\right).$$

*Démonstration.* Raisonnant comme pour la Proposition 1, on est ramené à ceci: supposons vérifiées les conditions (1) à (5) de la proposition; soit $T \in \hat{\Gamma}_4$, $T \notin A_1 \cup B_1 \cup C_1$; et montrons qu'il existe une $F \in \mathscr{S}(\Gamma_4)$ telle que $T(F) \neq 0$ et $T'(F) = 0$ pour $T' \in A_1$. Distinguons trois cas.

(A) $$T = U_{\lambda_0,\mu_0} \notin A_1.$$

Soient $\rho_1 \in \mathbf{R}$, $\theta \in \mathbf{R}$. Utilisons le Lemme 9 et la notation $E$ de ce lemme. Le point

$$\left(\rho_1 - \theta, \frac{1}{2}\frac{\mu_0}{\lambda_0} - \tfrac{1}{2}\lambda_0\theta^2, \lambda_0\theta, -\lambda_0\right)$$

n'appartient pas à $E$ (c'est immédiat), donc lui est extérieur (Lemme 9). Soit $G$ une fonction de $\mathscr{S}(\mathbf{R}^4)$ non nulle en ce point et nulle sur $E$. On a $G = \hat{F}_{234}$ pour $F \in \mathscr{S}(\mathbf{R}^4)$ bien choisie. Alors, compte tenu de (4), on a

$$U_{\lambda_0,\mu_0}(F) \neq 0,$$

et $U_{\lambda,\mu}(F) = 0$ pour $(\lambda, \mu) \in A_1$.

(B) $$T = V_{\lambda_0} \notin B_1.$$

D'après l'hypothèse (3) de la proposition, $(0, \lambda_0^2)$ n'est pas adhérent à $A_1$ dans $\mathbf{R}^2$. D'après le Lemme 9, quels que soient $\rho_1 \in \mathbf{R}$, $\theta \in \mathbf{R}$, le point $(\rho_1 - \theta, \lambda_0\theta, -\lambda_0, 0)$ est extérieur à $E$. Raisonnant comme dans (A), et compte tenu de (2) et (4), il existe $F \in \mathscr{S}(\mathbf{R}^4)$ telle que

$$V_{\lambda_0}(F) \neq 0$$

et $U_{\lambda,\mu}(F) = 0$ pour $(\lambda, \mu) \in A_1$.

(C) $T = W_{\lambda_0,\mu_0} \notin C_1$. D'après les hypothèses (4) et (5) de la proposition, et compte tenu du Lemme 9, $(-\lambda_0, -\mu_0, 0, 0)$ est extérieur à $E$. Raisonnant comme dans (A), il existe $F \in \mathscr{S}(\mathbf{R}^4)$ telle que $\hat{F}_{1234}$ soit nulle sur $E$ et non nulle en $(-\lambda_0, -\mu_0, 0, 0)$. Alors $\hat{F}_{234}$ est nulle sur $E$, donc $U_{\lambda,\mu}(F) = 0$ pour $(\lambda, \mu) \in A_1$ et

$$W_{\lambda_0,\mu_0}(F) \neq 0.$$

**4. Topologie de $\hat{\Gamma}_{5,1}$.** D'après (**1**, Proposition 5), $\hat{\Gamma}_{5,1}$ est une réunion de 2 sous-ensembles disjoints $A$ et $B$.

(1) $A$ est l'ensemble des $U_\lambda (\lambda \in \mathbf{R}, \lambda \neq 0)$; chaque $U_\lambda$ opère dans $L_{\mathbf{C}}^2(\mathbf{R}^2)$ et est définie par la formule

$$(U_\lambda(\gamma)f)(\theta_1, \theta_2) = \exp i\lambda(\rho_5 - \rho_2\theta_1 - \rho_4\theta_2)f(\theta_1 + \rho_1, \theta_2 + \rho_3)$$
$$(\gamma = (\rho_1, \ldots, \rho_5) \in \Gamma_{5,1}, f \in L_{\mathbf{C}}^2(\mathbf{R}^2), \theta_1, \theta_2 \in \mathbf{R}).$$

(2) $B$ est l'ensemble des $V_{\lambda,\mu,\nu,\tau}$ $(\lambda, \mu, \nu, \tau \in \mathbf{R})$; chaque $V_{\lambda,\mu,\nu,\tau}$ s'identifie à une fonction scalaire sur $\Gamma_{5,1}$, conformément à la relation

$$V_{\lambda,\mu,\nu,\tau}(\gamma) = \exp i(\lambda\rho_1 + \mu\rho_2 + \nu\rho_3 + \tau\rho_4).$$

D'après le Lemme 3, $B$ est canoniquement fermé dans $\hat{\Gamma}_{5,1}$, et la topologie induite sur $B$ par la topologie canonique de $\hat{\Gamma}_{5,1}$ est la topologie canonique de $\mathbf{R}^4 = \hat{\mathbf{R}}^4$.

On définit sur $\hat{\Gamma}_{5,1}$ une "topologie des paramètres" comme on l'a fait sur $\hat{\Gamma}_3$. Ceci posé, les calculs du paragraphe 2 se transposent sans difficultés et fournissent le résultat suivant:

PROPOSITION 3. *Les ensembles canoniquement fermés dans $\hat{\Gamma}_{5,1}$ sont les ensembles $A_1 \cup B_1$ $(A_1 \subset A, B_1 \subset B)$ possédant les propriétés suivantes:*

(1) *$A_1$ est fermé dans $A$ pour la topologie des paramètres;*

(2) *$B_1$ est fermé dans $B$ pour la topologie des paramètres;*

(3) *Si 0 est adhérent à $A_1$ dans $\mathbf{R}$, on a $B_1 = B$.*

**5. Topologie de $\hat{\Gamma}_{5,2}$.** D'après (**1**, Proposition 6), $\hat{\Gamma}_{5,2}$ est réunion de 2 sous-ensembles disjoints $A$, $B$:

(1) $A$ est l'ensemble des $U_{\lambda,\mu,\nu}$ ($\lambda, \mu, \nu \in \mathbf{R}, \lambda^2 + \mu^2 \neq 0$); chaque $U_{\lambda,\mu,\nu}$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$ et est définie par la formule

$$(U_{\lambda,\mu,\nu}(\gamma)f)(\theta) = \exp i\left[\nu\frac{\lambda\rho_3 - \mu\rho_2}{\lambda^2 + \mu^2} + \lambda(\rho_4 - \rho_2\theta) + \mu(\rho_5 - \rho_3\theta)\right]f(\theta + \rho_1)$$

$$(\gamma = (\rho_1, \ldots, \rho_5) \in \Gamma_{5,1}, f \in L_{\mathbf{C}}^2(\mathbf{R}), \theta \in \mathbf{R}).$$

(2) $B$ est l'ensemble des $V_{\lambda,\mu,\nu}$ ($\lambda, \mu, \nu \in \mathbf{R}$); chaque $V_{\lambda,\mu,\nu}$ s'identifie à une fonction scalaire sur $\Gamma_{5,2}$, conformément à la relation

$$V_{\lambda,\mu,\nu}(\gamma) = \exp i(\lambda\rho_1 + \mu\rho_2 + \nu\rho_3).$$

D'après le Lemme 3, $B$ est canoniquement fermé dans $\hat{\Gamma}_{5,2}$, et la topologie induite sur $B$ par la topologie canonique de $\hat{\Gamma}_{5,2}$ est celle de $\hat{\mathbf{R}}^3 = \mathbf{R}^3$.

On définit sur $\hat{\Gamma}_{5,2}$ une topologie des paramètres.

Avec les notations habituelles, on a

$$\begin{aligned}(5)\quad & (U_{\lambda,\mu,\nu}(F)f|g) \\ &= \int\cdots\int F(\rho_1, \ldots, \rho_5)\exp i\left[\nu\frac{\lambda\rho_3 - \mu\rho_2}{\lambda^2 + \mu^2} + \lambda(\rho_4 - \rho_2\theta) + \mu(\rho_5 - \rho_3\theta)\right] \\ &\qquad\qquad f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_5 d\theta \\ &= \int\cdots\int F(\rho_1 - \theta, \rho_2, \ldots, \rho_5)\exp i\left[\nu\frac{\lambda\rho_3 - \mu\rho_2}{\lambda^2 + \mu^2} + \lambda(\rho_4 - \rho_2\theta)\right. \\ &\qquad\qquad \left. + \mu(\rho_5 - \rho_3\theta)\right]f(\rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_5\, d\theta.\end{aligned}$$

Donc $U_{\lambda,\mu,\nu}(F)$ est défini par le noyau

$$\begin{aligned}(6)\quad & K_{\lambda,\mu,\nu}(\rho_1, \theta) \\ &= \int\cdots\int F(\rho_1 - \theta, \rho_2, \ldots, \rho_5)\exp i\left[\nu\frac{\lambda\rho_3 - \mu\rho_2}{\lambda^2 + \mu^2} + \lambda(\rho_4 - \rho_2\theta)\right. \\ &\qquad\qquad \left. + \mu(\rho_5 - \rho_3\theta)\right]d\rho_2 d\rho_3 d\rho_4 d\rho_5 \\ &= \hat{F}_{2345}\left(\rho_1 - \theta, \frac{\mu\nu}{\lambda^2 + \mu^2} + \lambda\theta, -\frac{\lambda\nu}{\lambda^2 + \mu^2} + \mu\theta, -\lambda, -\mu\right).\end{aligned}$$

LEMME 10. *Soit $F \in \mathscr{S}(\Gamma_{5,2})$. La fonction $T \to \|T(F)\|$ est continue sur $\hat{\Gamma}_{5,2}$ pour la topologie des paramètres.*

*Démonstration.* Les raisonnements sont analogues à ceux des Lemmes 5 et 7. Pour pouvoir utiliser le Lemme 4, il faut cette fois observer que

$$\begin{aligned}(7)\quad & (\rho_1 - \theta)^2 + \left(\frac{\mu\nu}{\lambda^2 + \mu^2} + \lambda\theta\right)^2 + \left(-\frac{\lambda\nu}{\lambda^2 + \mu^2} + \mu\theta\right)^2 + \lambda^2 + \mu^2 \\ &= (\rho_1 - \theta)^2 + (\lambda^2 + \mu^2)\theta^2 + \frac{\nu^2}{\lambda^2 + \mu^2} + \lambda^2 + \mu^2 \\ &> K(\rho_1{}^2 + \theta^2)\end{aligned}$$

pourvu que $\lambda^2 + \mu^2$ soit minoré par une constante $> 0$.

LEMME 11. *Soient $A_1 \subset A$, $B_1 \subset B$. On suppose $A_1 \cup B_1$ canoniquement fermé. Alors, si $(0,0,0)$ est adhérent à $A_1$ dans $\mathbf{R}^3$, $B_1$ contient tout point $(\alpha, \beta, \gamma)$ possédant la propriété suivante: il existe des suites $\beta_n$, $\gamma_n$, $\lambda_n$, $\mu_n$, $\nu_n$ de nombres réels tels que*

$$\beta_n \to \beta,\ \gamma_n \to \gamma,\ (\lambda_n, \mu_n, \nu_n) \to (0,0,0),\ (\lambda_n, \mu_n, \gamma_n) \in A_1,\ -\lambda_n\gamma_n + \mu_n\beta_n + \gamma_n = 0.$$

*Démonstration.* Soient $\alpha, \beta, \gamma, \beta_n, \gamma_n, \lambda_n, \mu_n, \nu_n$ avec les propriétés du lemme.

La condition $-\lambda_n\gamma_n + \mu_n\beta_n + \nu_n = 0$ exprime qu'on peut résoudre en $\theta_n$ le système:

$$\lambda_n\theta_n + \frac{\mu_n\nu_n}{\lambda_n^2 + \mu_n^2} = -\beta_n,\ \mu_n\theta_n - \frac{\lambda_n\nu_n}{\lambda_n^2 + \mu_n^2} = -\gamma_n.$$

Avec les notations habituelles, on a

$$(U_{\lambda_n,\mu_n,\nu_n}(F)\mathscr{T}(-\theta_n)f|\mathscr{T}(-\theta_n)g)$$
$$= \int \ldots \int F(\rho_1, \ldots, \rho_5) \exp(i\Delta) f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_5 d\theta$$

avec

$$\Delta = \nu_n \frac{\lambda_n\rho_3 - \mu_n\rho_2}{\lambda_n^2 + \mu_n^2} + \lambda_n\rho_4 - \lambda_n\rho_2(\theta + \theta_n) + \mu_n\rho_5 - \mu_n\rho_3(\theta + \theta_n)$$
$$= -\lambda_n\rho_2\theta - \mu_n\rho_3\theta + \lambda_n\rho_4 + \mu_n\rho_5 + \beta_n\rho_2 + \gamma_n\rho_3.$$

Quand $n \to +\infty$, l'intégrale tend vers

$$\int \ldots \int F(\rho_1, \ldots, \rho_5) \exp i(\beta\rho_2 + \gamma\rho_3) f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_5 d\theta$$
$$= \iint \hat{F}_{2345}(\rho_1, -\beta, -\gamma, 0, 0) f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 d\theta = (Vf|g)$$

où $V$ est, dans $L_{\mathbf{C}}^2(\mathbf{R})$, l'opérateur de convolution avec la fonction $\rho_1 \to \hat{F}_{2345}(-\rho_1, -\beta, -\gamma, 0, 0)$. Donc

$$\mathscr{T}(\theta_n)U_{\lambda_n,\mu_n,\nu_n}(F)\mathscr{T}(\theta_n)^{-1}$$

tend faiblement vers $V$ quand $n \to +\infty$, et par suite

$$\underline{\lim}_{n\to+\infty}||U_{\lambda_n,\mu_n,\nu_n}(F)|| \geqslant \sup_{\xi\in\mathbf{R}}|\hat{F}_{12345}(-\xi, -\beta, -\gamma, 0, 0)|$$
$$= \sup_{\xi\in\mathbf{R}}||V_{\xi,\beta,\gamma}(F)||.$$

Donc $V_{\alpha,\beta,\gamma} \in B_1$.

PROPOSITION 4. *Les ensembles canoniquement fermés dans $\Gamma_{5,2}$ sont les ensembles $A_1 \cup B_1$ ($A_1 \subset A$, $B_1 \subset B$) possédant les propriétés suivantes:*

(1) *$A_1$ est fermé dans $A$ pour la topologie des paramètres;*

(2) *$B_1$ est fermé dans $B$ pour la topologie des paramètres;*

(3) *Si $(0,0,0)$ est adhérent à $A_1$ dans $\mathbf{R}^3$, $B_1$ contient tout point $(\alpha, \beta, \gamma)$ tel qu'il existe des suites $\beta_n$, $\gamma_n$, $\lambda_n$, $\mu_n$, $\nu_n$, avec $\beta_n \to \beta$, $\gamma_n \to \gamma$, $(\lambda_n, \mu_n, \nu_n) \to (0,0,0)$, $(\lambda_n, \mu_n, \nu_n) \in A_1$, $-\lambda_n\gamma_n + \mu_n\beta_n + \nu_n = 0$.*

*Démonstration*. Raisonnant comme pour la Proposition 1, on est ramené à ceci: supposons vérifiées les conditions (1), (2), et (3) de la proposition; soit $T \in \hat{\Gamma}_{5,2}$, $T \notin A_1 \cup B_1$; et montrons qu'il existe une $F \in \mathscr{S}(\Gamma_{5,2})$ telle que $T(F) \neq 0$ et $T'(F) = 0$ pour $T' \in A_1$.

(A) Supposons

$$T = U_{\lambda_0,\mu_0,\nu_0} \notin A_1.$$

Soient $\rho_1 \in \mathbf{R}$, $\theta \in \mathbf{R}$. Soit $E$ l'ensemble des points de $\mathbf{R}^5$ de la forme

$$\left(t, \lambda\theta' + \frac{\mu\nu}{\lambda^2 + \mu^2}, \mu\theta' - \frac{\lambda\nu}{\lambda^2 + \mu^2}, -\lambda, -\mu\right),$$

où

$$t \in \mathbf{R}, \quad \theta' \in \mathbf{R}, \quad (\lambda, \mu, \nu) \in A_1.$$

Comme $A_1$ est fermé dans $A$ pour la topologie des paramètres, il est facile de voir que $E$ est fermé dans l'ensemble des points de $\mathbf{R}^5$ dont les deux dernières coordonnées ne sont pas nulles simultanément. Donc

$$\left(\rho_1 - \theta, \lambda_0\theta + \frac{\mu_0\nu_0}{\lambda_0^2 + \mu_0^2}, \mu_0\theta - \frac{\lambda_0\nu_0}{\lambda_0^2 + \mu_0^2}, -\lambda_0, -\mu_0\right)$$

est extérieur à $E$. D'où l'existence d'une $F \in \mathscr{S}(\Gamma_{5,2})$ telle que $\hat{F}_{2345}$ soit non nulle en ce point et nulle sur $E$. Alors,

$$U_{\lambda_0,\mu_0,\nu_0}(F) \neq 0,$$

et $U_{\lambda,\mu,\nu}(F) = 0$ pour $(\lambda, \mu, \nu) \in A_1$.

(B) Supposons $T = V_{\lambda_0,\mu_0,\nu_0} \notin B_1$. Alors, $(-\lambda_0, -\mu_0, -\nu_0, 0, 0)$ n'est pas adhérent à $E$ dans $\mathbf{R}^5$. En effet, supposons qu'il existe des suites $t_n, \theta_n, \lambda_n, \mu_n, \nu_n$ avec

$$t_n \to -\lambda_0, \lambda_n\theta_n + \frac{\mu_n\nu_n}{\lambda_n^2 + \mu_n^2} \to -\mu_0, \mu_n\theta_n - \frac{\lambda_n\nu_n}{\lambda_n^2 + \mu_n^2} \to -\nu_0,$$
$$\lambda_n \to 0, \mu_n \to 0, (\lambda_n, \mu_n, \nu_n) \in A_1.$$

Posons

$$\beta_n = -\lambda_n\theta_n - \frac{\mu_n\nu_n}{\lambda_n^2 + \mu_n^2}, \gamma_n = -\mu_n\theta_n + \frac{\lambda_n\nu_n}{\lambda_n^2 + \mu_n^2}.$$

On a $-\lambda_n\gamma_n + \mu_n\beta_n + \nu_n = 0$, et $\nu_n = \lambda_n\gamma_n - \mu_n\beta_n \to 0$. D'après l'hypothèse (3) de la proposition, on a donc $(\lambda_0, \mu_0, \nu_0) \in B_1$, ce qui est contradictoire. Ceci posé, il existe une $F \in \mathscr{S}(\Gamma_{5,2})$ telle que $\hat{F}_{12345}$ soit nulle sur $E$ et non nulle en $(-\lambda_0, -\mu_0, -\nu_0, 0, 0)$. Alors $\hat{F}_{2345}$ est nulle sur $E$, donc $U_{\lambda,\mu,\nu}(F) = 0$ pour $(\lambda, \mu, \nu) \in A_1$, et

$$V_{\lambda_0,\mu_0,\nu_0}(F) \neq 0.$$

**6. Topologie de $\hat{\Gamma}_{5,3}$.** D'après (**1**, Proposition 7), $\hat{\Gamma}_{5,3}$ est réunion de 3 sous-ensembles disjoints $A$, $B$, $C$:

(1) $A$ est l'ensemble des $U_\lambda(\lambda \in \mathbf{R}, \lambda \neq 0)$; chaque $U_\lambda$ opère dans $L_{\mathbf{C}}^2(\mathbf{R}^2)$ et est définie par la formule

$$(U_\lambda(\gamma)f)(\theta_1, \theta_2) = \exp i\lambda(\rho_5 - \rho_4\theta_1 + \tfrac{1}{2}\rho_2\theta_1{}^2 - \rho_3\theta_2)f(\theta_1 + \rho_1, \theta_2 + \rho_2)$$

$$(\gamma = (\rho_1, \ldots, \rho_5) \in \Gamma_{5,3}, f \in L_{\mathbf{C}}^2(\mathbf{R}^2), \theta_1, \theta_2 \in \mathbf{R}).$$

(2) $B$ est l'ensemble des $V_{\lambda,\mu}(\lambda, \mu \in \mathbf{R}, \lambda \neq 0)$; chaque $V_{\lambda,\mu}$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$ et est définie par la formule

$$(V_{\lambda,\mu}(\gamma)f)(\theta) = \exp i(\lambda\rho_4 - \lambda\rho_3\theta + \mu\rho_3)f(\theta + \rho_1).$$

(3) $C$ est l'ensemble des $W_{\lambda,\mu,\nu}(\lambda, \mu, \nu \in \mathbf{R})$; chaque $W_{\lambda,\mu,\nu}$ s'identifie à une fonction scalaire sur $\Gamma_{5,3}$, conformément à la relation

$$W_{\lambda,\mu,\nu}(\gamma) = \exp i(\lambda\rho_1 + \mu\rho_2 + \nu\rho_3).$$

D'après le Lemme 3, $B \cup C$ est canoniquement fermé dans $\hat{\Gamma}_{5,3}$, et la topologie induite sur $B \cup C$ par la topologie canonique de $\hat{\Gamma}_{5,3}$ est la topologie canonique de $(\Gamma_3 \times \mathbf{R})^- = \hat{\Gamma}_3 \times \mathbf{R}$.

On définit sur $\hat{\Gamma}_{5,3}$ une topologie des paramètres.

Avec les notations habituelles, on a

(8) $(U_\lambda(F)f|g) =$

$$\int \cdots \int F(\rho_1, \ldots, \rho_5) \exp i\lambda(\rho_5 - \rho_4\theta_1 + \tfrac{1}{2}\rho_2\theta_1^2 - \rho_3\theta_2)f(\theta_1 + \rho_1, \theta_2 + \rho_2)$$
$$\overline{g(\theta_1, \theta_2)}d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2 =$$
$$\int \cdots \int F(\rho_1 - \theta_1, \rho_2 - \theta_2, \rho_3, \rho_4, \rho_5) \exp(i\lambda A)f(\rho_1, \rho_2)\overline{g(\theta_1, \theta_2)}d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2$$

où

$$A = \rho_5 - \rho_4\theta_1 + \tfrac{1}{2}\rho_2\theta_1^2 - \tfrac{1}{2}\theta_2\theta_1^2 - \rho_3\theta_2.$$

Donc $U_\lambda(F)$ est défini par le noyau

(9) $K_\lambda(\rho_1, \rho_2, \theta_1, \theta_2)$

$$= \iiint F(\rho_1 - \theta_1, \rho_2 - \theta_2, \rho_3, \rho_4, \rho_5) \exp(i\lambda A)d\rho_3 d\rho_4 d\rho_5$$
$$= \exp \tfrac{1}{2}i\lambda(\rho_2 - \theta_2)\theta_1^2 \,.\, \hat{F}_{345}(\rho_1 - \theta_1, \rho_2 - \theta_2, \lambda\theta_2, \lambda\theta_1, -\lambda).$$

LEMME 12. *Soit $F \in \mathscr{S}(\Gamma_{5,3})$. La fonction $T \to \|T(F)\|$ est continue sur $\hat{\Gamma}_{5,3}$ pour la topologie des paramètres.*

*Démonstration.* D'après le Lemme 4, la fonction $(\rho_1, \rho_2, \theta_1, \theta_2) \to \hat{F}_{345}(\rho_1 - \theta_1, \rho_2 - \theta_2, \lambda\theta_2, \lambda\theta_1, -\lambda)$ est un élément de $L_{\mathbf{C}}^2(\mathbf{R}^4)$ qui dépend continûment de $\lambda \in \mathbf{R} - \{0\}$ pour la topologie forte. D'autre part, l'opérateur unitaire de multiplication par $\exp \frac{1}{2}i\lambda(\rho_2 - \theta_2)\theta_1{}^2$ dans $L_{\mathbf{C}}^2(\mathbf{R}^4)$ dépend continûment de $\lambda$ pour la topologie forte. Donc l'application $\lambda \to K_\lambda$ de $\mathbf{R} - \{0\}$ dans $L_{\mathbf{C}}^2(\mathbf{R}^4)$ est fortement continue. On achève alors comme pour le Lemme 5.

LEMME 13. *Soient* $A_1 \subset A$, $B_1 \subset B$, $C_1 \subset C$. *On suppose* $A_1 \cup B_1 \cup C_1$ *canoniquement fermé. Si* 0 *est adhérent à* $A_1$ *dans* $\mathbf{R}$, *on a* $B_1 = B$, $C_1 = C$.

*Démonstration*. Soient $\beta_1, \beta_2 \in \mathbf{R}$. Avec les notations habituelles, on a

$$\left(U_\lambda(F)\mathscr{T}\left(-\frac{\beta_1}{\lambda}, -\frac{\beta_2}{\lambda}\right)\mathscr{M}\left(\exp\left(-\tfrac{1}{2}i\frac{\beta_1^2}{\lambda}\theta_2\right)\right)f \,\middle|\, \mathscr{T}\left(-\frac{\beta_1}{\lambda}, -\frac{\beta_2}{\lambda}\right)\right.$$
$$\left.\mathscr{M}\left(\exp\left(-\tfrac{1}{2}i\frac{\beta_1^2}{\lambda}\theta_2\right)\right)g\right)$$
$$= \int \cdots \int F(\rho_1, \ldots, \rho_5) \exp i\lambda(\rho_5 - \rho_4\theta_1 + \tfrac{1}{2}\rho_2\theta_1^2 - \rho_3\theta_2)$$
$$\exp - \tfrac{1}{2}i\frac{\beta_1^2}{\lambda}\left(\theta_2 - \frac{\beta_2}{\lambda} + \rho_2\right)$$
$$f\left(\theta_1 - \frac{\beta_1}{\lambda} + \rho_1, \theta_2 - \frac{\beta_2}{\lambda} + \rho_2\right) \exp \tfrac{1}{2}i\frac{\beta_1^2}{\lambda}\left(\theta_2 - \frac{\beta_2}{\lambda}\right)\overline{g\left(\theta_1 - \frac{\beta_1}{\lambda}, \theta_2 - \frac{\beta_2}{\lambda}\right)}$$
$$d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2$$
$$= \int \cdots \int F(\rho_1, \ldots, \rho_5) \exp(i\Delta) f(\theta_1 + \rho_1, \theta_2 + \rho_2)\overline{g(\theta_1, \theta_2)}d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2$$

avec

$$\Delta = \lambda\rho_5 - \lambda\rho_4\left(\theta_1 + \frac{\beta_1}{\lambda}\right) + \tfrac{1}{2}\lambda\rho_2\left(\theta_1 + \frac{\beta_1}{\lambda}\right)^2 - \lambda\rho_3\left(\theta_2 + \frac{\beta_2}{\lambda}\right) - \tfrac{1}{2}\frac{\beta_1{}^2}{\lambda}\rho_2$$
$$= \lambda\rho_5 - \lambda\rho_4\theta_1 + \tfrac{1}{2}\lambda\rho_2\theta_1{}^2 - \lambda\rho_3\theta_2 - \beta_1\rho_4 + \beta_1\rho_2\theta_1 - \beta_2\rho_3.$$

Quand $\lambda \to 0$, l'intégrale tend vers

$$\int \cdots \int F(\rho_1, \ldots, \rho_5) \exp i(-\beta_2\rho_3 - \beta_1\rho_4 + \beta_1\rho_2\theta_1) f(\theta_1 + \rho_1, \theta_2 + \rho_2)$$
$$\overline{g(\theta_1, \theta_2)}d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2.$$

D'autre part, on peut identifier canoniquement $L_{\mathbf{C}}{}^2(\mathbf{R}^2)$ à l'espace des fonctions de carré intégrable sur $\mathbf{R}$ pour la mesure de Lebesgue et à valeurs dans $L_{\mathbf{C}}{}^2(\mathbf{R})$ c'est-à-dire encore à l'intégrale hilbertienne

$$\int_{\mathbf{R}}^{\oplus} \mathfrak{H}_t \, dt$$

où, pour tout $t$, $\mathfrak{H}_t = L_{\mathbf{C}}{}^2(\mathbf{R})$; dans cette identification, l'élément $(\theta_1, \theta_2) \to f(\theta_1, \theta_2)$ de $L_{\mathbf{C}}{}^2(\mathbf{R}^2)$ correspond au champ de vecteurs $\theta_2 \to f_{\theta_2}$, où $f_{\theta_2}$ est l'élément de $L_{\mathbf{C}}{}^2(\mathbf{R})$ défini (pour presque tout $\theta_2$) par

$$f_{\theta_2}(\theta_1) = f(\theta_1, \theta_2).$$

Supposons $\beta_1 \neq 0$. Pour chaque $\theta_2 \in \mathbf{R}$, considérons l'opérateur

$$\mathcal{T}\left(-\frac{\theta_2}{\beta_1}\right) V_{-\beta_1,-\beta_2}(F)\, \mathcal{T}\left(\frac{\theta_2}{\beta_1}\right)$$

dans

$$\mathfrak{H}_{\theta_2} = L^2_{\mathbf{C}}(\mathbf{R}).$$

Cet opérateur a une norme indépendante de $\theta_2$ et dépend continûment $\theta_2$ pour la topologie forte. L'application

$$\theta_2 \to \mathcal{T}\left(-\frac{\theta_2}{\beta_1}\right) V_{-\beta_1,-\beta_2}(F)\, \mathcal{T}\left(\frac{\theta_2}{\beta_1}\right)$$

est donc un champ continu borné d'opérateurs, et définit dans

$$\int_{\mathbf{R}}^{\oplus} \mathfrak{H}_{\theta_2} d\theta_2 = L^2_{\mathbf{C}}(\mathbf{R}^2)$$

un opérateur $S$, de norme

$$\| V_{-\beta_1,\,-\beta_2}(F) \|,$$

tel que

$$(Sf|g) = \int \left( V_{-\beta_1,-\beta_2}(F)\, \mathcal{T}\left(\frac{\theta_2}{\beta_1}\right) f_{\theta_2}, \mathcal{T} \,\middle|\, \left(\frac{\theta_2}{\beta_1}\right) g_{\theta_2} \right) d\theta_2$$

$$= \int \ldots \int F(\rho_1, \ldots, \rho_5) \exp i(-\beta_2\rho_3 - \beta_1\rho_4 + \beta_1\rho_2\theta_1)\, f\left(\theta_1 + \rho_1 + \frac{\theta_2}{\beta_1}, \theta_2\right)$$

$$\overline{g\left(\theta_1 + \frac{\theta_2}{\beta_1}, \theta_2\right)} d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2$$

$$= \int \ldots \int F(\rho_1, \ldots, \rho_5) \exp i(-\beta_2\rho_3 - \beta_1\rho_4 + \beta_1\rho_2\theta_1) \exp(-i\rho_2\theta_2)$$

$$f(\theta_1 + \rho_1, \theta_2)\overline{g(\theta_1, \theta_2)} d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2.$$

Désignons par $\mathcal{F}_2$ la transformation de Fourier par rapport à la 2e variable dans $L_{\mathbf{C}}^2(\mathbf{R}^2)$. D'après la formule de Plancherel, on a

$$\int \exp(-i\rho_2\theta_2) f(\theta_1 + \rho_1, \theta_2)\overline{g(\theta_1, \theta_2)} d\theta_2$$

$$= \frac{1}{2\pi} \int (\mathcal{F}_2 f)(\theta_1 + \rho_1, \theta_2 + \rho_2)\overline{(\mathcal{F}_2 g)(\theta_1, \theta_2)}\, d\theta_2.$$

Donc

$$(\mathcal{F}_2 S \mathcal{F}_2^{-1} f|g) = 2\pi (S\mathcal{F}_2^{-1} f | \mathcal{F}_2^{-1} g)$$

$$= \int \ldots \int F(\rho_1, \ldots, \rho_5) \exp i(-\beta_2\rho_3 - \beta_1\rho_4 + \beta_1\rho_2\theta_1) f(\theta_1 + \rho_1, \theta_2 + \rho_2)$$

$$\overline{g(\theta_1, \theta_2)} d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2.$$

En définitive, quand $\lambda \to 0$, l'opérateur

$$\mathscr{M}\left(\exp\left(\tfrac{1}{3}i\frac{\beta_1^2}{\lambda}\theta_2\right)\right)\mathscr{T}\left(\frac{\beta_1}{\lambda},\frac{\beta_2}{\lambda}\right)U_\lambda(F)\mathscr{T}\left(\frac{\beta_1}{\lambda},\frac{\beta_2}{\lambda}\right)^{-1}\mathscr{M}\left(\exp\left(\tfrac{1}{3}i\frac{\beta_1^2}{\lambda}\theta_2\right)\right)^{-1}$$

tend faiblement vers $\mathscr{F}_2 S \mathscr{F}_2^{-1}$. Par suite,

$$\underline{\lim}_{\lambda\to 0}\|U_\lambda(F)\| \geqslant \|S\| = \|V_{-\beta_1,-\beta_2}(F)\|.$$

Donc $V_{\lambda,\mu} \in B_1$ quels que soient $\lambda$ et $\mu$, ce qui prouve que $B_1 = B$. Comme $B_1$ est partout dense dans $B \cup C$ pour la topologie canonique d'après la Proposition 1, on a aussi $C_1 = C$.

PROPOSITION 5. *Les ensembles canoniquement fermés dans $\hat{\Gamma}_{5,3}$ sont les ensembles $A_1 \cup B_1 \cup C_1$ $(A_1 \subset A, B_1 \subset B, C_1 \subset C)$ possédant les propriétés suivantes:*

(1) *$A_1$ est fermé dans $A$ pour la topologie des paramètres;*

(2) *$B_1 \cup C_1$ est canoniquement fermé dans $B \cup C = \hat{\Gamma}_3 \times \mathbf{R}$;*

(3) *Si 0 est adhérent à $A_1$ dans $\mathbf{R}$, on a $B_1 = B$, $C_1 = C$.*

*Démonstration.* Supposons remplies les conditions (1), (2), et (3) de la proposition. Soit $T \in \hat{\Gamma}_{5,3}$ avec $T \notin A_1 \cup B_1 \cup C_1$. Il s'agit de construire une $F \in \mathscr{S}(\Gamma_{5,3})$ telle que $T(F) \neq 0$ et $T'(F) = 0$ pour $T' \in A_1$.

(A) Supposons $T = U_{\lambda_0} \notin A_1$. Il existe $F \in \mathscr{S}(\Gamma_{5,3})$ telle que $\hat{F}_{345}$ soit nulle lorsque la 5e variable appartient à $A_1$ et non nulle lorsque la 5e variable a pour valeur $\lambda_0$. Alors, compte tenu de (9), on a $U_{\lambda_0}(F) \neq 0$ et $U_\lambda(F) = 0$ pour $\lambda \in A_1$.

(B) Supposons $T = V_{\lambda_0,\mu_0} \notin B_1$. D'après la condition (3) de la proposition, 0 est non adhérent à $A_1$ dans $\mathbf{R}$. Donc il existe $F \in \mathscr{S}(\Gamma_{5,3})$ telle que $\hat{F}_{2345}(\rho_1 - \theta, \lambda_0\theta, -\mu_0, -\lambda_0, 0) \neq 0$ pour certaines valeurs de $\rho_1$ et $\theta$, et telle que $\hat{F}_{2345}$ (donc aussi $\hat{F}_{345}$) soit nulle lorsque la 5e variable appartient à $A_1$. La première condition entraîne facilement que $V_{\lambda_0,\mu_0}(F) \neq 0$; la deuxième entraîne que $U_\lambda(F) = 0$ pour $\lambda \in A_1$.

(C) Si $T = W_{\lambda_0,\mu_0,\nu_0} \in C_1$, le raisonnement est analogue à celui de (B).

**7. Topologie de $\hat{\Gamma}_{5,5}$.** D'après (**1**, Proposition 9), $\hat{\Gamma}_{5,5}$ est réunion de 3 sous-ensembles disjoints $A$, $B$, $C$;

(1) $A$ est l'ensemble des $U_{\lambda,\mu,\nu,\rho}$ $(\lambda, \mu, \nu, \rho \in \mathbf{R}, \mu^3 - \nu^2 + \lambda^2\rho = 0, \lambda^2 + \mu^2 + \nu^2 \neq 0)$; chaque $U_{\lambda,\mu,\nu,\rho}$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$ et est définie par la formule

$$(U_{\lambda,\mu,\nu,\rho}(\gamma)f)(\theta) =$$

$$\exp i\left(-\frac{1}{3}\frac{\nu}{\lambda^2}\rho_2 - \frac{1}{2}\frac{\mu}{\lambda}(\rho_3 - \rho_2\theta) + \lambda(\rho_5 - \rho_4\theta + \tfrac{1}{2}\rho_3\theta^2 - \tfrac{1}{6}\rho_2\theta^3)\right)f(\theta + \rho_1)$$

si $\lambda \neq 0$, et par la formule

$$(U_{\lambda,\mu,\nu,\rho}(\gamma)f)(\theta) = \exp i\left(-\frac{1}{6}\frac{\rho}{\nu}\rho_2 - \frac{\nu}{\mu}(\rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2)\right)f(\theta + \rho_1)$$

si $\lambda = 0$.

$$(\gamma = (\rho_1, \ldots, \rho_5) \in \Gamma_{5,5}, f \in L^2_{\mathbf{C}}(\mathbf{R}), \theta \in \mathbf{R}).$$

(2) $B$ est l'ensemble des $V_\lambda$ ($\lambda \in \mathbf{R}, \lambda \neq 0$); chaque $V_\lambda$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$ et est définie par la formule

$$(V_\lambda(\gamma)f)(\theta) = \exp i\lambda(\rho_5 - \rho_2\theta)f(\theta + \rho_1).$$

(3) $C$ est l'ensemble des $W_{\lambda,\mu}$ ($\lambda, \mu \in \mathbf{R}$); chaque $W_{\lambda,\mu}$ s'identifie à une fonction scalaire sur $\Gamma_{5,5}$ conformément à la relation

$$W_{\lambda,\mu}(\gamma) = \exp i(\lambda\rho_1 + \mu\rho_2).$$

D'après le Lemme 3, $B \cup C$ est canoniquement fermé dans $\hat{\Gamma}_{5,5}$, et la topologie induite sur $B \cup C$ par la topologie canonique de $\hat{\Gamma}_{5,5}$ est la topologie de $\hat{\Gamma}_3$.

On définit sur $\hat{\Gamma}_{5,5}$ une topologie des paramètres.

Avec les notations habituelles, on a, pour $\lambda \neq 0$

$$\begin{aligned}(10)\quad &(U_{\lambda,\mu,\nu,\rho}(F)f|g)\\ &= \int \ldots \int F(\rho_1, \ldots, \rho_5) \exp(iA)f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_5 d\theta\\ &= \int \ldots \int F(\rho_1 - \theta, \rho_2, \ldots, \rho_5) \exp(iA)f(\rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_5 d\theta,\end{aligned}$$

avec

$$A = -\frac{1}{3}\frac{\nu}{\lambda^2}\rho_2 - \frac{1}{2}\frac{\mu}{\lambda}(\rho_3 - \rho_2\theta) + \lambda(\rho_5 - \rho_4\theta + \tfrac{1}{2}\rho_3\theta^2 - \tfrac{1}{6}\rho_2\theta^3).$$

Donc $U_{\lambda,\mu,\nu,\rho}(F)$ est défini, pour $\lambda \neq 0$, par le noyau

$$\begin{aligned}(11)\quad &\int \ldots \int F(\rho_1 - \theta, \rho_2, \ldots, \rho_5) \exp(iA)d\rho_2 d\rho_3 d\rho_4 d\rho_5\\ &\qquad = \hat{F}_{2345}\left(\rho_1 - \theta, \frac{1}{3}\frac{\nu}{\lambda^2} - \frac{1}{2}\frac{\mu}{\lambda}\theta + \tfrac{1}{6}\lambda\theta^3, \frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2, \lambda\theta, -\lambda\right).\end{aligned}$$

Pour $\lambda = 0$, on a

$$\begin{aligned}(12)\quad &(U_{\lambda,\mu,\nu,\rho}(F)f|g)\\ &= \int \ldots \int F(\rho_1, \ldots, \rho_5) \exp i\left(-\frac{1}{6}\frac{\rho}{\nu}\rho_2 - \frac{\nu}{\mu}(\rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2)\right)\\ &\qquad\qquad f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_5 d\theta\\ &= \int \ldots \int F(\rho_1 - \theta, \rho_2, \ldots, \rho_5) \exp i\left(-\frac{1}{6}\frac{\rho}{\nu}\rho_2 - \frac{\nu}{\mu}(\rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2)\right)\\ &\qquad\qquad f(\rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_5 d\theta.\end{aligned}$$

Donc $U_{\lambda,\mu,\nu,\rho}(F)$ est défini, pour $\lambda = 0$, par le noyau

$$(13)\quad \int \dots \int F(\rho_1 - \theta, \rho_2, \dots, \rho_5) \exp i\left(-\frac{1}{6}\frac{\rho}{\nu}\rho_2 - \frac{\nu}{\mu}(\rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2)\right) d\rho_2 \dots d\rho_5$$
$$= \hat{F}_{2345}\left(\rho_1 - \theta, \frac{1}{6}\frac{\rho}{\nu} + \frac{1}{2}\frac{\nu}{\mu}\theta^2, -\frac{\nu}{\mu}\theta, \frac{\nu}{\mu}, 0\right).$$

Nous nous contenterons de démontrer le résultat suivant:

Proposition 6. *La topologie induite sur $A$ par la topologie canonique est la topologie des paramètres.*

*Démonstration*. Soit $A_1$ une partie de $A$ canoniquement fermée dans $A$, et montrons que $A_1$ est fermée dans $A$ pour la topologie des paramètres. Soit $(\lambda_n, \mu_n, \nu_n, \rho_n)$ une suite de points de $A_1$ telle que $\lambda_n, \mu_n, \nu_n, \rho_n$ aient des limites finies $\lambda, \mu, \nu, \rho$, où $\lambda, \mu, \nu$ ne sont pas simultanément nuls. Il s'agit de montrer que $(\lambda, \mu, \nu, \rho) \in A_1$.

Si $\lambda \neq 0$, la formule (10) montre que

$$(U_{\lambda_n,\mu_n,\nu_n,\rho_n}(F)f|g) \to (U_{\lambda,\mu,\nu,\rho}(F)f|g),$$

donc que

$$U_{\lambda_n,\mu_n,\nu_n,\rho_n}(F)$$

tend faiblement vers $U_{\lambda,\mu,\nu,\rho}(F)$, donc que

$$\underline{\lim}_{n\to+\infty} \| U_{\lambda_n,\mu_n,\nu_n,\rho_n}(F) \| \geq \| U_{\lambda,\mu,\nu,\rho}(F) \|.$$

Alors $U_{\lambda,\mu,\nu,\rho}$ est canoniquement adhérent à $A_1$ (Lemme 1), donc $U_{\lambda,\mu,\nu,\rho} \in A_1$. On raisonne de même si $\lambda = 0$ et si $\lambda_n = 0$ pour une infinité de valeurs de $n$.

Supposons maintenant $\lambda = 0$, et $\lambda_n \neq 0$ pour une infinité de valeurs de $n$, donc, en changeant de notations, pour tout $n$. On a $\mu^3 - \nu^2 = 0$, donc $\mu > 0$. Faisons d'abord l'hypothèse que $\nu < 0$, d'où $\mu(\mu)^{\frac{1}{2}} = -\nu$. On a $\mu_n > 0$ pour $n$ assez grand, et

$$\left(U_{\lambda_n,\mu_n,\nu_n,\rho_n}(F)\mathscr{T}\left(\frac{\mu_n^{\frac{1}{2}}}{\lambda_n}\right)f \,\middle|\, \mathscr{T}\left(\frac{\mu_n^{\frac{1}{2}}}{\lambda_n}\right)g\right)$$
$$= \int \dots \int F(\rho_1, \dots, \rho_5) \exp(i\Delta) f(\theta + \rho_1)\overline{g(\theta)}\, d\rho_1 \dots d\rho_5 d\theta$$

avec

$$\Delta = -\frac{1}{3}\frac{\nu_n}{\lambda_n^2}\rho_2 - \frac{1}{2}\frac{\mu_n}{\lambda_n}\rho_3 + \frac{1}{2}\frac{\mu_n}{\lambda_n}\rho_2\left(\theta - \frac{\mu_n^{\frac{1}{2}}}{\lambda_n}\right) + \lambda_n\rho_5 - \lambda_n\rho_4\left(\theta - \frac{\mu_n^{\frac{1}{2}}}{\lambda_n}\right)$$
$$+ \tfrac{1}{2}\lambda_n\rho_3\left(\theta - \frac{\mu_n^{\frac{1}{2}}}{\lambda_n}\right)^2 - \tfrac{1}{6}\lambda_n\rho_2\left(\theta - \frac{\mu_n^{\frac{1}{2}}}{\lambda_n}\right)^3$$
$$= -\frac{1}{3}\frac{\nu_n + \mu_n\mu_n^{\frac{1}{2}}}{\lambda_n^2}\rho_2 + \lambda_n(\rho_5 - \rho_4\theta + \tfrac{1}{2}\rho_3\theta^2 - \tfrac{1}{6}\rho_2\theta^3) + \mu_n^{\frac{1}{2}}(\rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2)$$
$$= -\frac{1}{3}\frac{\rho_n}{\nu_n - \mu_n\mu_n^{\frac{1}{2}}}\rho_2 + \lambda_n(\rho_5 - \rho_4\theta + \tfrac{1}{2}\rho_3\theta^2 - \tfrac{1}{6}\rho_2\theta^3) + \mu_n^{\frac{1}{2}}(\rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2).$$

Donc, quand $n \to +\infty$,

$$\left(U_{\lambda_n, \mu_n, \nu_n, \rho_n}(F)\, \mathscr{T}\left(\frac{\mu_n^{\frac{1}{3}}}{\lambda_n}\right) f \,\middle|\, \mathscr{T}\left(\frac{\mu_n^{\frac{1}{3}}}{\lambda_n}\right) g\right) \to$$

$$\int \ldots \int F(\rho_1, \ldots, \rho_5) \exp i\left(-\frac{1}{6}\frac{\rho}{\nu}\rho_2 - \frac{\nu}{\mu}(\rho_4 - \rho_3\theta + \tfrac{1}{2}\rho_2\theta^2)\right)$$

$$f(\theta + \rho_1)\overline{g(\theta)}d\rho_1 \ldots d\rho_5 d\theta = (U_{\lambda, \mu, \nu, \rho}(F)f|g),$$

et le raisonnement s'achève comme plus haut. Enfin, si $\nu > 0$, il suffit de changer partout $\mu_n^{\frac{1}{3}}$ en $-\mu_n^{\frac{1}{3}}$ dans ce qui précède.

Pour établir que, réciproquement, une partie de $A$ fermée pour la topologie des paramètres est canoniquement fermée dans $A$, nous aurons besoin du lemme suivant.

LEMME 14. *Soit $A_1$ une partie de $A$ fermée pour la topologie des paramètres. Soit $E_1$ l'ensemble des points de $\mathbf{R}^4$ de la forme*

$$\left(\frac{1}{3}\frac{\nu}{\lambda^2} - \frac{1}{2}\frac{\mu}{\lambda}\theta + \tfrac{1}{6}\lambda\theta^3, \quad \frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2, \quad \lambda\theta, -\lambda\right)$$

*où $\theta \in \mathbf{R}$, $(\lambda, \mu, \nu, \rho) \in A_1$, $\lambda \neq 0$. Soit $E_2$ l'ensemble des points de $\mathbf{R}^4$ de la forme*

$$\left(\frac{1}{6}\frac{\rho}{\nu} + \frac{1}{2}\frac{\nu}{\mu}\theta^2, \quad -\frac{\nu}{\mu}\theta, \quad \frac{\nu}{\mu}, 0\right)$$

*où $\theta \in \mathbf{R}$, $(0, \mu, \nu, \rho) \in A_1$. Alors, $E_1 \cup E_2$ est fermé dans $\mathbf{R}^4 - \Omega$, en désignant par $\Omega$ l'ensemble des points de $\mathbf{R}^4$ dont les 2 dernières coordonnées sont nulles.*

*Démonstration.* (A) Supposons d'abord que

$$\frac{1}{3}\frac{\nu_n}{\lambda_n^2} - \frac{1}{2}\frac{\mu_n}{\lambda_n}\theta_n + \tfrac{1}{6}\lambda_n\theta_n^3 \to \xi_1, \frac{1}{2}\frac{\mu_n}{\lambda_n} - \tfrac{1}{2}\lambda_n\theta_n^2 \to \xi_2, \lambda_n\theta_n \to \xi_3, -\lambda_n \to \xi_4,$$

où $(\lambda_n, \mu_n, \nu_n, \rho_n) \in A_1$, $\lambda_n \neq 0$, et où $\xi_1, \xi_2, \xi_3, \xi_4$ sont des nombres réels finis tels que $\xi_3$ *et* $\xi_4$ ne soient pas nuls tous les deux. Montrons que $(\xi_1, \xi_2, \xi_3, \xi_4) \in E_1 \cup E_2$.

Si $\xi_4 \neq 0$, $\theta_n$, donc, $\mu_n/\lambda_n$ donc $\nu_n/\lambda_n^2$, donc $\mu_n$ et $\nu_n$, ont des limites finies, donc

$$\rho_n = \frac{\nu_n^2 - \mu_n^3}{\lambda_n^2}$$

a une limite finie. Soient $\lambda, \mu, \nu, \rho, \theta$ les limites de $\lambda_n, \mu_n, \nu_n, \rho_n, \theta_n$. On a $(\lambda, \mu, \nu, \rho) \in A_1$, et

$$\frac{1}{3}\frac{\nu}{\lambda^2} - \frac{1}{2}\frac{\mu}{\lambda}\theta + \tfrac{1}{6}\lambda\theta^3 = \xi_1, \frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2 = \xi_2, \lambda\theta = \xi_3, -\lambda = \xi_4,$$

donc $(\xi_1, \xi_2, \xi_3, \xi_4) \in E_1$.

Si $\xi_4 = 0$ (donc $\xi_3 \neq 0$), on a $\mu_n - (\lambda_n\theta_n)^2 \to 0$, donc $\mu_n \to \xi_3^2 > 0$, et $2\nu_n - 3\mu_n\lambda_n\theta_n + (\lambda_n\theta_n)^3 \to 0$, donc $\nu_n \to \xi_3^3$. Par ailleurs,

$$9\left(\frac{1}{3}\frac{\nu_n}{\lambda_n^2} - \frac{1}{2}\frac{\mu_n}{\lambda_n}\theta_n + \tfrac{1}{6}\lambda_n\theta_n^3\right)^2 \lambda_n^2 - 8\left(\frac{1}{2}\frac{\mu_n}{\lambda_n} - \tfrac{1}{2}\lambda_n\theta_n^2\right)^3 \lambda_n \to 0$$

$$18\left(\frac{1}{3}\frac{\nu_n}{\lambda_n^2} - \frac{1}{2}\frac{\mu_n}{\lambda_n}\theta_n + \tfrac{1}{6}\lambda_n\theta_n^3\right)\left(\frac{1}{2}\frac{\mu_n}{\lambda_n} - \tfrac{1}{2}\lambda_n\theta_n^2\right) \lambda_n^2\theta_n \to 0$$

$$3\lambda_n^2\theta_n^2\left[2\left(\frac{1}{3}\frac{\nu_n}{\lambda_n^2} - \frac{1}{2}\frac{\mu_n}{\lambda_n}\theta_n + \tfrac{1}{6}\lambda_n\theta_n^3\right)\lambda_n\theta_n - \left(\frac{1}{2}\frac{\mu_n}{\lambda_n} - \tfrac{1}{2}\lambda_n\theta_n^2\right)^2\right]$$
$$\to 3\xi_3^2(2\xi_1\xi_3 - \xi_2^2).$$

Additionnant, on trouve que

$$\frac{\nu_n^2 - \mu_n^3}{\lambda_n^2} = \rho_n$$

tend vers $3\xi_3^2(2\xi_1\xi_3 - \xi_2^2)$. Posons $\mu = \xi_3^2$, $\nu = \xi_3^3$, $\rho = 3\xi_3^2(2\xi_1\xi_3 - \xi_2^2)$. Ce qui précède prouve que $(0, \mu, \nu, \rho) \in A_1$; posant de plus $\theta = -\xi_2/\xi_3$, on constate que

$$\frac{1}{6}\frac{\rho}{\nu} + \frac{1}{2}\frac{\nu}{\mu}\theta^2 = \xi_1, \qquad -\frac{\nu}{\mu}\theta = \xi_2, \qquad \frac{\nu}{\mu} = \xi_3$$

donc $(\xi_1, \xi_2, \xi_3, \xi_4) \in E_2$.

(B) Supposons que

$$\frac{1}{6}\frac{\rho_n}{\nu_n} + \frac{1}{2}\frac{\nu_n}{\mu_n}\theta_n^2 \to \xi_1, \qquad -\frac{\nu_n}{\mu_n}\theta_n \to \xi_2, \qquad \frac{\nu_n}{\mu_n} \to \xi_3$$

où $(0, \mu_n, \nu_n, \rho_n) \in A_1$, et où $\xi_1$, $\xi_2$, $\xi_3$ sont des nombres réels finis tels que $\xi_3 \neq 0$. Comme $\nu_n^2 = \mu_n^3$, on a $\nu_n/\mu_n = \nu_n^{\frac{1}{3}}$, donc $\mu_n$ et $\nu_n$ ont des limites finies $\mu = \xi_3^2$ et $\nu = \xi_3^3$. Alors, $\theta_n$ et $\rho_n$ ont des limites finies $\theta$ et $\rho$. On a $(0, \mu, \nu, \rho) \in A_1$, et

$$\frac{1}{6}\frac{\rho}{\nu} + \frac{1}{2}\frac{\nu}{\mu}\theta^2 = \xi_1, \qquad -\frac{\nu}{\mu}\theta = \xi_2, \qquad \frac{\nu}{\mu} = \xi_3,$$

donc $(\xi_1, \xi_2, \xi_3, 0) \in E_2$. Ceci achève la démonstration du lemme.

Revenons à la démonstration de la Proposition 6. Soit $A_1$ une partie de $A$ fermée pour la topologie des paramètres. Soit $(\lambda_0, \mu_0, \nu_0, \rho_0)$ un point de $A$ n'appartenant pas à $A_1$. Soient $\rho_1, \theta$ deux nombres réels. Conservons les notations du Lemme 14. Si $\lambda_0 \neq 0$, le point

$$\left(\frac{1}{3}\frac{\nu_0}{\lambda_0^2} - \frac{1}{2}\frac{\mu_0}{\lambda_0}\theta + \tfrac{1}{6}\lambda_0\theta^3, \qquad \frac{1}{2}\frac{\mu_0}{\lambda_0} - \tfrac{1}{2}\lambda_0\theta^2, \qquad \lambda_0\theta, \qquad -\lambda_0\right)$$

n'appartient pas à $E_1 \cup E_2$, donc est extérieur à $E_1 \cup E_2$. Si $\lambda_0 = 0$, le point

$$\left(\frac{1}{6}\frac{\rho_0}{\nu_0} + \frac{1}{2}\frac{\nu_0}{\mu_0}\theta^2, \qquad -\frac{\nu_0}{\mu_0}\theta, \qquad \frac{\nu_0}{\mu_0}, 0\right)$$

n'appartient pas à $E_1 \cup E_2$, donc est extérieur à $E_1 \cup E_2$. Dans les deux cas, il existe une $F \in \mathscr{S}(\Gamma_{5,5})$ telle que $\hat{F}_{2345}$ soit nulle sur $\mathbf{R} \times (E_1 \cup E_2)$ et non nulle au point

$$\left(\rho_1 - \theta, \quad \frac{1}{3}\frac{\nu_0}{\lambda_0^2} - \frac{1}{2}\frac{\mu_0}{\lambda_0}\theta + \tfrac{1}{6}\lambda_0\theta^3, \quad \frac{1}{2}\frac{\mu_0}{\lambda_0} - \tfrac{1}{2}\lambda_0\theta^2, \quad \lambda_0\theta, \quad -\lambda_0\right)$$

(resp. au point

$$\left(\rho_1 - \theta, \quad \frac{1}{6}\frac{\rho_0}{\nu_0} + \frac{1}{2}\frac{\nu_0}{\mu_0}\theta^2, \quad -\frac{\nu_0}{\mu_0}\theta, \quad \frac{\nu_0}{\mu_0}, 0\right)$$

si $\lambda_0 = 0$). Alors,

$$U_{\lambda_0,\mu_0,\nu_0,\rho_0}(F) \neq 0,$$

et $U_{\lambda,\mu,\nu,\rho}(F) = 0$ pour $(\lambda, \mu, \nu, \rho) \in A_1$. Donc $U_{\lambda_0,\mu_0,\nu_0,\rho_0}$ est extérieur à $A_1$ pour la topologie canonique. Ceci prouve que $A_1$ est canoniquement fermé, et achève la démonstration de la proposition 6.

*Remarque.* La mesure de Plancherel sur $A \subset \hat{\Gamma}_{5,5}$ est défiine par la forme différentiell $w = \lambda^{-2}d\lambda d\mu d\nu$ (**1**, Proposition 9). Comme $2\lambda^{-2}d\lambda d\mu d\nu = \nu^{-1}d\lambda d\mu d\rho$, $w$ est régulière sur tout $A$, contrairement à ce qui est dit dans (**1**), p. 322, l. 18–21.

**8. Topologie de $\hat{\Gamma}_{5,6}$.** D'après (**1**, Proposition 10), $\hat{\Gamma}_{5,6}$ est réunion de 4 sous-ensembles disjoints $A$, $B$, $C$, $D$:

(1) $A$ est l'ensemble des $U_\lambda (\lambda \in \mathbf{R}, \lambda \neq 0)$; chaque $U_\lambda$ opère dans $L_{\mathbf{C}}^2(\mathbf{R}^2)$ et est définie par la formule

$$\begin{aligned}&(U_\lambda(\gamma)f)(\theta_1, \theta_2)\\ &= \exp i\lambda(\rho_6 - \rho_4\theta_1 + \tfrac{1}{2}\rho_2^2\theta_1 + \tfrac{1}{2}\rho_3\theta_1^2 - \tfrac{1}{6}\rho_2\theta_1^3 - \rho_3\theta_2 + \rho_2\theta_1\theta_2)f(\theta_1 + \rho_1, \theta_2 + \rho_2)\\ &\qquad (\gamma = (\rho_1, \ldots, \rho_6) \in \Gamma_{5,6}, \quad f \in L_{\mathbf{C}}^2(\mathbf{R}^2), \quad \theta_1, \theta_2 \in \mathbf{R}).\end{aligned}$$

(2) $B$ est l'ensemble des $V_{\lambda,\mu}$ $(\lambda, \mu \in \mathbf{R}, \lambda \neq 0)$; chaque $V_{\lambda,\mu}$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$ et est définie par la formule

$$(V_{\lambda,\mu}(\gamma)f)(\theta) = \exp i\left(-\frac{1}{2}\frac{\mu}{\lambda}\rho_2 + \lambda\rho_4 - \lambda\rho_3\theta + \tfrac{1}{2}\lambda\rho_2\theta^2\right)f(\theta + \rho_1).$$

(3) $C$ est l'ensemble des $W_\lambda$ $(\lambda \in \mathbf{R}, \lambda \neq 0)$; chaque $W_\lambda$ opère dans $L_{\mathbf{C}}^2(\mathbf{R})$ et est définie par la formule

$$(W_\lambda(\gamma)f)(\theta) = \exp i\lambda(\rho_3 - \rho_2\theta)f(\theta + \rho_1).$$

(4) $D$ est l'ensemble des $X_{\lambda,\mu}$ $(\lambda, \mu \in \mathbf{R})$; chaque $X_{\lambda,\mu}$ s'identifie à une fonction scalaire sur $\Gamma_{5,6}$ conformément à la relation

$$X_{\lambda,\mu}(\gamma) = \exp i(\lambda\rho_1 + \mu\rho_2).$$

D'après le Lemme 3, $B \cup C \cup D$ est canoniquement fermé dans $\hat{\Gamma}_{5,6}$ et la topologie induite sur $B \cup C \cup D$ par la topologie canonique de $\hat{\Gamma}_{5,6}$ est la topologie canonique de $\hat{\Gamma}_4$.

On définit sur $\Gamma_{5,6}$ une topologie des paramètres.

Avec les notations habituelles, on a

(14) $(U_\lambda(F)f|g)$

$$= \int \dots \int F(\rho_1, \dots, \rho_5) \exp i\lambda(\rho_5 - \rho_4\theta_1 + \tfrac{1}{2}\rho_2^2\theta_1 + \tfrac{1}{2}\rho_3\theta_1^2 - \tfrac{1}{6}\rho_2\theta_1^3 - \rho_3\theta_2 + \rho_2\theta_1\theta_2) f(\theta_1 + \rho_1, \theta_2 + \rho_2)\overline{g(\theta_1, \theta_2)}d\rho_1 \dots d\rho_5 d\theta_1 d\theta_2$$

$$= \int \dots \int F(\rho_1 - \theta_1, \rho_2 - \theta_2, \rho_3, \rho_4, \rho_5) \exp(i\lambda A) f(\rho_1, \rho_2)\overline{g(\theta_1, \theta_2)}d\rho_1 \dots d\rho_5 d\theta_1 d\theta_2$$

avec

$$A = \rho_5 - \rho_4\theta_1 + \tfrac{1}{2}(\rho_2 - \theta_2)^2\theta_1 + \tfrac{1}{2}\rho_3\theta_1^2 - \tfrac{1}{6}(\rho_2 - \theta_2)\theta_1^3 - \rho_3\theta_2 + (\rho_2 - \theta_2)\theta_1\theta_2.$$

Donc $U_\lambda(F)$ est définie par le noyau

(15) $K_\lambda(\rho_1, \rho_2, \theta_1, \theta_2)$

$$= \iiint F(\rho_1 - \theta_1, \rho_2 - \theta_2, \rho_3, \rho_4, \rho_5) \exp(i\lambda A) d\rho_3 d\rho_4 d\rho_5$$

$$= \exp i\lambda(\tfrac{1}{2}(\rho_2 - \theta_2)^2\theta_1 - \tfrac{1}{6}(\rho_2 - \theta_2)\theta_1^3 + (\rho_2 - \theta_2)\theta_1\theta_2) \hat{F}_{345}(\rho_1 - \theta_1, \rho_2 - \theta_2, \lambda\theta_2 - \tfrac{1}{2}\lambda\theta_1^2, \lambda\theta_1, -\lambda).$$

Lemme 15. *Soit $F \in \mathscr{S}(\Gamma_{5,6})$. La fonction $T \to \|T(F)\|$ est continue sur $\Gamma_{5,6}$ pour la topologie des paramètres.*

Démonstration analogue à celle du Lemme 12.

Lemme 16. *Soient $A_1 \subset A$, $B_1 \subset B$, $C_1 \subset C$, $D_1 \subset D$. On suppose $A_1 \cup B_1 \cup C_1 \cup D_1$ canoniquement fermé. Si 0 est adhérent à $A_1$ dans $\mathbf{R}$, on a $B_1 = B$, $C_1 = C$, $D_1 = D$.*

*Démonstration.* Soit $\beta_1 \in \mathbf{R} - \{0\}$. Avec les notations habituelles, on a

$$\Big(U_\lambda(F)\mathscr{T}\Big(-\frac{\beta_1}{\lambda}, -\frac{\beta_1^2}{2\lambda^2}\Big)\mathscr{M}\Big(\exp -i\Big(\frac{1}{3}\frac{\beta_1^3}{\lambda^2}\theta_2 + \tfrac{1}{2}\beta_1\theta_2^2\Big)\Big)f \Big| \mathscr{T}\Big(-\frac{\beta_1}{\lambda}, -\frac{\beta_1^2}{2\lambda^2}\Big)\mathscr{M}\Big(\exp -i\Big(\frac{1}{3}\frac{\beta_1^3}{\lambda^2}\theta_2 + \tfrac{1}{2}\beta_1\theta_2^2\Big)\Big)g\Big)$$

$$= \int \dots \int F(\rho_1, \dots, \rho_5) \exp(i\Delta) \exp\Big[-i\Big(\frac{1}{3}\frac{\beta_1^3}{\lambda^2}\Big(\theta_2 - \frac{\beta_1^2}{2\lambda^2} + \rho_2\Big) + \tfrac{1}{2}\beta_1\Big(\theta_2 - \frac{\beta_1^2}{2\lambda^2} + \rho_2\Big)^2\Big)\Big]$$

$$\exp\Big[i\Big(\frac{1}{3}\frac{\beta_1^3}{\lambda^2}\Big(\theta_2 - \frac{\beta_1^2}{2\lambda^2}\Big) + \tfrac{1}{2}\beta_1\Big(\theta_2 - \frac{\beta_1^2}{2\lambda^2}\Big)^2\Big]f\Big(\theta_1 - \frac{\beta_1}{\lambda} + \rho_1, \theta_2 - \frac{\beta_1^2}{2\lambda^2} + \rho_2\Big) \overline{g\Big(\theta_1 - \frac{\beta_1}{\lambda}, \theta_2 - \frac{\beta_1^2}{2\lambda^2}\Big)} d\rho_1 \dots d\rho_5 d\theta_1 d\theta_2$$

$$= \int \cdots \int F(\rho_1, \ldots, \rho_5) \exp(i\Delta) \exp\left[-i\left(\frac{1}{3}\frac{\beta_1^3}{\lambda^2}\rho_2 + \beta_1\left(\theta_2 - \frac{\beta_1^2}{2\lambda^2}\right)\rho_2 + \tfrac{1}{2}\beta_1\rho_2^2\right)\right] f\left(\theta_1 - \frac{\beta_1}{\lambda} + \rho_1, \theta_2 - \frac{\beta_1^2}{2\lambda^2} + \rho_2\right) \overline{g\left(\theta_1 - \frac{\beta_1}{\lambda}, \theta_2 - \frac{\beta_1^2}{2\lambda^2}\right)} \, d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2$$

$$= \int \cdots \int F(\rho_1, \ldots, \rho_5) \exp(i\Delta') f(\theta_1 + \rho_1, \theta_2 + \rho_2) \overline{g(\theta_1, \theta_2)} d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2$$

avec

$$\begin{aligned}\Delta' = {} & \lambda\rho_5 - \lambda\rho_4\left(\theta_1 + \frac{\beta_1}{\lambda}\right) + \tfrac{1}{2}\lambda\rho_2^2\left(\theta_1 + \frac{\beta_1}{\lambda}\right) + \tfrac{1}{2}\lambda\rho_3\left(\theta_1 + \frac{\beta_1}{\lambda}\right)^2 - \tfrac{1}{6}\lambda\rho_2\left(\theta_1 + \frac{\beta_1}{\lambda}\right)^3 \\ & - \lambda\rho_3\left(\theta_2 + \frac{\beta_1^2}{2\lambda^2}\right) + \lambda\rho_2\left(\theta_1 + \frac{\beta_1}{\lambda}\right)\left(\theta_2 + \frac{\beta_1^2}{2\lambda^2}\right) - \frac{1}{3}\frac{\beta_1^3}{\lambda^2}\rho_2 - \beta_1\rho_2\theta_2 - \tfrac{1}{2}\beta_1\rho_2^2 \\ = {} & \lambda(\rho_5 - \rho_4\theta_1 + \tfrac{1}{2}\rho_2^2\theta_1 + \tfrac{1}{2}\rho_3\theta_1^2 - \tfrac{1}{6}\rho_2\theta_1^3 - \rho_3\theta_2 + \rho_2\theta_1\theta_2) - \beta_1\rho_4 + \beta_1\rho_3\theta_1 - \tfrac{1}{2}\beta_1\rho_2\theta_1^2.\end{aligned}$$

Donc, quand $\lambda \to 0$, l'intégrale étudiée a pour limite

$$\int \cdots \int F(\rho_1, \ldots, \rho_5) \exp i(-\beta_1\rho_4 + \beta_1\rho_3\theta_1 - \tfrac{1}{2}\beta_1\rho_2\theta_1^2)\, f(\theta_1 + \rho_1, \theta_2 + \rho_2) \overline{g(\theta_1, \theta_2)}\, d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2.$$

Identifions, comme au paragraphe 6, l'espace $L_{\mathbf{C}}^2(\mathbf{R}^2)$ à l'intégrale hilbertienne

$$\int^{\oplus} \mathfrak{H}_{\theta_2} d\theta_2,$$

où

$$\mathfrak{H}_{\theta_2} = L_{\mathbf{C}}^2(\mathbf{R})$$

pour tout $\theta_2 \in \mathbf{R}$. Le champ d'opérateurs

$$\theta_2 \to V_{-\beta_1, -2\beta_1\theta_2}(F)$$

est fortement continu (et même beaucoup mieux!) et il définit dans $L_{\mathbf{C}}^2(\mathbf{R}^2)$ un opérateur $S$ tel que

$$\begin{aligned}(Sf|g) &= \int (V_{-\beta_1, -2\beta_1\theta_2}(F) f_{\theta_2} | g_{\theta_2})\, d\theta_2 \\ &= \int \cdots \int F(\rho_1, \ldots, \rho_5) \exp i(-\rho_2\theta_2 - \beta_1\rho_4 + \beta_1\rho_3\theta_1 - \tfrac{1}{2}\beta_1\rho_2\theta_1^2) \\ &\qquad f(\theta_1 + \rho_1, \theta_2)\overline{g(\theta_1, \theta_2)} d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2.\end{aligned}$$

Désignant encore par $\mathscr{F}_2$ la transformation de Fourier par rapport à la 2e variable dans $L_{\mathbf{C}}^2(\mathbf{R}^2)$, on obtient

$$(\mathscr{F}_2 S \mathscr{F}_2^{-1} f|g) = \int \ldots \int F(\rho_1, \ldots, \rho_5) \exp i(-\beta_1\rho_4 + \beta_1\rho_2\theta_1 - \tfrac{1}{2}\beta_1\rho_2\theta_1^2)$$

$$f(\theta_1 + \rho_1, \theta_2 + \rho_2)\overline{g(\theta_1, \theta_2)} d\rho_1 \ldots d\rho_5 d\theta_1 d\theta_2.$$

En définitive, quand $\lambda \to 0$, l'opérateur

$$\mathscr{M}\left(\exp i\left(\frac{1}{3}\frac{\beta_1^3}{\lambda^2}\theta_2 + \tfrac{1}{2}\beta_1\theta_2^2\right)\right)\mathscr{T}\left(\frac{\beta_1}{\lambda}, \frac{\beta_1^2}{2\lambda^2}\right) U_\lambda(F) \mathscr{T}\left(\frac{\beta_1}{\lambda}, \frac{\beta_1^2}{2\lambda^2}\right)^{-1}$$

$$\mathscr{M}\left(\exp i\left(\frac{1}{3}\frac{\beta_1^3}{\lambda^2}\theta_2 + \tfrac{1}{2}\beta_1\theta_2^2\right)\right)^{-1}$$

tend faiblement vers $\mathscr{F}_2 S \mathscr{F}_2^{-1}$. Par suite

$$\underline{\lim}_{\lambda\to 0} ||U_\lambda(F)|| \geqslant \sup_{\theta_2 \in \mathbf{R}} ||V_{-\beta_1, -2\beta_1\theta_2}(F)|| = \sup_{\mu \in \mathbf{R}} ||V_{-\beta_1, \mu}(F)||.$$

Donc $V_{-\beta_1, \mu} \in B_1$. Comme $\beta_1$ et $\mu$ sont arbitraires ($\beta_1 \neq 0$), on a $B_1 = B$. D'après la Proposition 2, $B$ est partout dense dans $B \cup C \cup D$ pour la topologie canonique, donc $C_1 = C$, $D_1 = D$.

Proposition 7. *Les ensembles canoniquement fermés de $\hat{\Gamma}_{5,6}$ sont les ensembles $A_1 \cup B_1 \cup C_1 \cup D_1$ ($A_1 \subset A$, $B_1 \subset B$, $C_1 \subset C$, $D_1 \subset D$) possédant les propriétés suivantes:*

(1) *$A_1$ est fermé dans $A$ pour la topologie des paramètres;*

(2) *$B_1 \cup C_1 \cup D_1$ est canoniquement fermé dans $B \cup C \cup D = \hat{\Gamma}_4$;*

(3) *Si 0 est adhérent à $A_1$ dans $\mathbf{R}$, on a $B_1 = B$, $C_1 = C$, $D_1 = D$.*

*Démonstration*. Supposons remplies les conditions (1), (2), et (3) de la proposition. Soit $T \in \hat{\Gamma}_{5,6}$ avec $T \notin A_1 \cup B_1 \cup C_1 \cup D_1$. Il s'agit de construire une $F \in \mathscr{S}(\Gamma_{5,6})$ telle que $T(F) \neq 0$ et $T'(F) = 0$ pour $T' \in A_1$.

(A) Supposons $T = U_{\lambda_0} \notin A_1$. Il existe $F \in \mathscr{S}(\Gamma_{5,6})$ telle que $\hat{F}_{345}$ soit nulle lorsque la 5e variable appartient à $A_1$ et non nulle lorsque la 5e variable a pour valeur $\lambda_0$. Alors $U_{\lambda_0}(F) \neq 0$ et $U_\lambda(F) = 0$ pour $\lambda \in A_1$.

(B) Supposons $T = V_{\lambda_0, \mu_0} \notin B_1$. Alors, d'après la condition (3) de la proposition, 0 est non adhérent à $A_1$ dans $\mathbf{R}$. Donc il existe $F \in \mathscr{S}(\hat{\Gamma}_{5,6})$ telle que $\hat{F}_{2345}$, et par suite $\hat{F}_{345}$, soient nulles lorsque la 5e variable appartient à $A_1$, et telle que la fonction

$$(\rho_1, \theta) \to \hat{F}_{2345}\left(\rho_1 - \theta, \frac{1}{2}\frac{\mu_0}{\lambda_0} - \tfrac{1}{2}\lambda_0\theta^2, \lambda_0\theta, -\lambda_0, 0\right)$$

ne soit pas identiquement nulle. Alors $V_{\lambda_0, \mu_0}(F) \neq 0$, et $U_\lambda(F) = 0$ pour $\lambda \in A_1$.

(C) On raisonne de manière analogue pour $T = W_{\lambda_1} \notin C_1$ ou $T = X_{\lambda_0, \mu_0} \notin D_1$.

## 9. Etude des caractères.

Proposition 8. *Soit $\Gamma_4'$ le sous-groupe de $\Gamma_4$ d'équation $\rho_1 = 0$. Le caractère de $U_{\lambda, \mu}$ est une mesure concentrée sur $\Gamma_4'$. Sa restriction à $\Gamma_4'$ est définie par la densité*

$$(16) \qquad (1 \pm i)\left(\frac{\pi}{|\lambda\rho_2|}\right)^{\frac{1}{2}} \exp \tfrac{1}{2}i\left(\lambda\, \frac{2\rho_2\rho_4 - \rho_3^2}{\rho_2} - \frac{\mu}{\lambda}\,\rho_2\right)$$

*où il faut prendre le signe + pour $\lambda\rho_2 > 0$, et le signe − pour $\lambda\rho_2 < 0$.*

*Démonstration.* On a, pour $F \in \mathscr{S}(\Gamma_4)$,

$$\mathrm{tr}(U_{\lambda,\mu}(F)) = \int \hat{F}_{234}\left(0, \frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2, \lambda\theta, -\lambda\right) d\theta.$$

Donc le caractère de $U_{\lambda,\mu}$ est l'extension à $\Gamma_4$ d'une distribution définie sur $\Gamma_4'$. Identifiant $\Gamma_4'$ à $\mathbf{R}^3$ grâce au système de coordonnées $(\rho_2, \rho_3, \rho_4)$ on voit que cette distribution est la transformée de Fourier $\mathscr{F}m$ de la mesure $m$ à croissance lente définie par la formule

$$m(f) = \int f\left(\frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2, \lambda\theta, -\lambda\right) d\theta$$

($f$, fonction continue à support compact dans $\mathbf{R}^3$). Soit $m_T$ la mesure à support compact définie par la formule

$$m_T(f) = \int_{-T}^{T} f\left(\frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2, \lambda\theta, -\lambda\right) d\theta.$$

Quand $T \to +\infty$, $m_T$ tend vers $m$ au sens des distributions tempérées. Donc $\mathscr{F}m_T$ tend vers $\mathscr{F}m$ au sens des distributions tempérées. Or, $\mathscr{F}m_T$ est la fonction

$$(17) \qquad (\rho_2, \rho_3, \rho_4) \to \int_{-T}^{+T} \exp -i\left[\left(\frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2\right)\rho_2 + \lambda\theta\rho_3 - \lambda\rho_4\right] d\theta.$$

Nous allons vérifier que, quand $T \to +\infty$, cette fonction converge simplement vers la fonction (16), en restant majorée en module par une fonction localement intégrable fixe. On en conclura que la fonction (17) tend vers la fonction (16) au sens des distributions, donc que la fonction (16) est la distribution $\mathscr{F}m$ cherchée.

Supposons $\lambda\rho_2 > 0$. Dans l'intégrale (17), faisons le changement de variables $(\tfrac{1}{2}\lambda\rho_2)^{\frac{1}{2}}(\theta - \rho_3/\rho_2) = \zeta$. Elle devient

$$\int_{(\frac{\lambda\rho_2}{2})^{\frac{1}{2}}(-T-\rho_3/\rho_2)}^{(\frac{\lambda\rho_2}{2})^{\frac{1}{2}}(T-\rho_3/\rho_2)} \exp i\left[\zeta^2 - \tfrac{1}{2}\lambda\frac{\rho_3^2}{\rho_2} - \frac{1}{2}\frac{\mu}{\lambda}\rho_2 + \lambda\rho_4\right]\left(\frac{2}{\lambda\rho_2}\right)^{\frac{1}{2}} d\zeta$$

$$= \left(\frac{2}{\lambda\rho_2}\right)^{\frac{1}{2}} \exp\frac{i}{2}\left(\lambda\,\frac{2\rho_2\rho_4 - \rho_3^2}{\rho_2} - \frac{\mu}{\lambda}\rho_2\right) \int_{(\frac{\lambda\rho_2}{2})^{\frac{1}{2}}(-T-\rho_3/\rho_2)}^{(\frac{\lambda\rho_2}{2})^{\frac{1}{2}}(T-\rho_3/\rho_2)} \exp(i\zeta^2)\, d\zeta.$$

Or, un calcul élémentaire montre que

$$\left|\int_a^b \exp(i\zeta^2)\, d\zeta\right| < 4$$

quels que soient $a$ et $b$. Et, quand $T \to +\infty$, l'intégrale tend vers

$$\int_{-\infty}^{+\infty} \exp(i\zeta^2)\, d\zeta = \sqrt{\frac{\pi}{2}}\,(1 + i).$$

La proposition est donc vérifiée pour $\lambda\rho_2 > 0$. On procède de façon analogue pour $\lambda\rho_2 < 0$.

PROPOSITION 9. *Soit $\Gamma_{5,5}'$ le sous-groupe de $\Gamma_{5,5}$ d'équation $\rho_1 = 0$. Pour $\lambda \neq 0$, le caractère de $U_{\lambda,\mu,\nu,\rho}$ est une mesure concentrée sur $\Gamma_{5,5}'$. Sa restriction à $\Gamma_{5,5}'$ est définie par la densité d suivante; on pose*

$$\alpha = \frac{1}{2}\left(\frac{6}{\lambda\rho_2}\right)^{\frac{1}{3}}\left(\lambda\,\frac{2\rho_2\rho_4 - \rho_3^2}{\rho_2} - \frac{\mu}{\lambda}\,\rho_2\right)$$

$$d = \pm\left(\frac{6}{\lambda\rho_2}\right)^{\frac{1}{3}} \exp\frac{i}{3}\left(\frac{3\rho_2^2\rho_5 - 3\rho_2\rho_3\rho_4 + \rho_3^3}{\rho_2^2} - \frac{\nu}{\lambda^2}\,\rho_2\right)\Delta(\rho_2, \rho_3, \rho_4)$$

*où il faut prendre le signe + pour $\lambda\rho_2 > 0$, le signe − pour $\lambda\rho_2 < 0$, et où*

$$\Delta = \tfrac{2}{3}\alpha^{\frac{1}{2}} K_{1/3}\left(\frac{2\alpha^{3/2}}{3\sqrt{3}}\right) \qquad \textit{pour} \qquad \alpha > 0$$

$$\Delta = \frac{2\pi}{3}\,(\tfrac{1}{3}\alpha)^{\frac{1}{2}}\,[J_{1/3}(\tfrac{2}{3}\alpha(\tfrac{1}{3}\alpha)^{\frac{1}{2}}) + J_{-1/3}(\tfrac{2}{3}\alpha(\tfrac{1}{3}\alpha)^{\frac{1}{2}})] \qquad \textit{pour} \qquad \alpha < 0.$$

($J_{1/3}, J_{-1/3}, K_{1/3}$ *sont les notations classiques pour les fonctions de Bessel.*)

*Démonstration.* On a, pour $F \in \mathscr{S}(\Gamma_{5,5})$ et $\lambda \neq 0$,

$$\operatorname{tr}(U_{\lambda,\mu,\nu,\rho}(F)) = \int \hat{F}_{2345}\left(0, \frac{1}{3}\frac{\nu}{\lambda^2} - \frac{1}{2}\frac{\mu}{\lambda}\theta + \tfrac{1}{6}\lambda\theta^3, \frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2, \lambda\theta, -\lambda\right) d\theta.$$

Donc le caractère de $U_{\lambda,\mu,\nu,\rho}$ et l'extension à $\Gamma_{5,5}$ d'une distribution définie sur $\Gamma_{5,5}'$. Identifiant $\Gamma_{5,5}'$ à $\mathbf{R}^4$ grâce au système de coordonnées $(\rho_2, \rho_3, \rho_4, \rho_5)$, on voit que cette distribution est la transformée de Fourier $\mathscr{F}m$ de la mesure $m$ à croissance lente définie par la formule

$$m(f) = \int f\left(\frac{1}{3}\frac{\nu}{\lambda^2} - \frac{1}{2}\frac{\mu}{\lambda}\theta + \tfrac{1}{6}\lambda\theta^3, \frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2, \lambda\theta, -\lambda\right) d\theta$$

($f$, fonction continue à support compact dans $\mathbf{R}^4$). Procédant comme pour la Proposition 8, on est amené à considérer l'intégrale

$$(18) \quad \int_{-T}^{+T} \exp -i\left[\left(\frac{1}{3}\frac{\nu}{\lambda^2} - \frac{1}{2}\frac{\mu}{\lambda}\theta + \tfrac{1}{6}\lambda\theta^3\right)\rho_2 + \left(\frac{1}{2}\frac{\mu}{\lambda} - \tfrac{1}{2}\lambda\theta^2\right)\rho_3 + \lambda\theta\rho_4 - \lambda\rho_5\right] d\theta.$$

Supposons $\rho_2 \neq 0$. Faisant, dans cette intégrale, le changement de variable

$$(\tfrac{1}{6}\lambda\rho_2)^{\frac{1}{3}}\left(\theta - \frac{\rho_3}{\rho_2}\right) = \zeta,$$

elle devient

$$\left(\frac{6}{\lambda\rho_2}\right)^{\frac{1}{3}} \exp\frac{i}{3}\left(\lambda\,\frac{3\rho_2^2\rho_5 - 3\rho_2\rho_3\rho_4 + \rho_3^3}{\rho_2^2} - \frac{\nu}{\lambda^2}\,\rho_2\right) \int_u^v \exp -i\left[\zeta^3 + \frac{1}{2}\left(\frac{6}{\lambda\rho_2}\right)^{\frac{1}{3}}\right.$$

$$\left.\left(\lambda\,\frac{2\rho_2\rho_4 - \rho_3^2}{\rho_2} - \frac{\mu}{\lambda}\,\rho_2\right)\zeta\right] d\zeta$$

avec

$$u = (\tfrac{1}{6}\lambda\rho_2)^{\frac{1}{3}}\left(-T - \frac{\rho_3}{\rho_2}\right) \qquad v = (\tfrac{1}{6}\lambda\rho_2)^{\frac{1}{3}}\left(T - \frac{\rho_3}{\rho_2}\right).$$

Une majoration élémentaire montre que

$$\left|\int_a^b \exp -i(\zeta^3 + \alpha\zeta)d\zeta\right| < 10$$

quels que soient $a$, $b$, $\alpha$. Donc $\mathscr{F}m$ est la fonction

$$\pm\left(\frac{6}{\lambda\rho_2}\right)^{\frac{1}{3}} \exp\frac{i}{3}\left(\lambda\,\frac{3\rho_2^2\rho_5 - 3\rho_2\rho_3\rho_4 + \rho_3^3}{\rho_2^2} - \frac{\nu}{\lambda^2}\,\rho_2\right)\int_{-\infty}^{+\infty}\exp -i(\zeta^3 + \alpha\zeta)\,d\zeta$$

où il faut prendre le signe + pour $\lambda\rho_2 > 0$, le signe − pour $\lambda\rho_2 < 0$. Or, on a (intégrale d'Airy)

$$\int_{-\infty}^{+\infty}\exp i(\zeta^3 + \alpha\zeta)\,d\zeta = \begin{cases} \tfrac{2}{3}\alpha^{\frac{1}{2}}K_{1/3}\,(2(\tfrac{1}{3}\alpha)^{3/2})\ (\alpha > 0) \\ 2\pi/3\ (\tfrac{1}{3}\alpha)^{\frac{1}{2}}[J_{1/3}\,(2(\tfrac{1}{3}\alpha)^{3/2}) + J_{-1/3}\,(2(\tfrac{1}{3}\alpha)^{3/2})](\alpha<0). \end{cases}$$

D'où la proposition.

Je dois à Delsarte la remarque que la limite de l'intégrale (18), quand $T \to +\infty$, peut se calculer explicitement.

## Bibliographie


1. J. Dixmier, *Sur les représentations unitaires des groupes de Lie nilpotents.* III, Can. J. Math., *10* (1958), 321–348.
2. ———, *Sur les représentations unitaires des groupes de Lie nilpotents.* V, Bull. Soc. Math. France, *87* (1959), 65–79.
3. J. M. G. Fell, *The dual spaces of C*-algebras, à paraître aux* Trans. Amer. Math. Soc.
4. R. Godement, *Les fonctions de type positif et la théorie des groupes,* Trans. Amer. Math. Soc., *63* (1948), 1–84.


*Institut Henri Poincaré*

Wiley BOOKS

## MATHEMATICAL METHODS FOR DIGITAL COMPUTERS

*Edited by* ANTHONY RALSTON, *The Bell Telephone Laboratories; and* HERBERT S. WILF, *The University of Illinois.* Treats numerical analysis strictly from the point of view of the digital computer—covering the mathematical analysis of techniques which are useful in digital computation, the actual coding procedure, the medium of calculation procedure, and other topics. *1960. 293 pages. $9.00.*

## INTERMEDIATE ALGEBRA

*By* ROY DUBISCH, STEVEN J. BRYANT, *and* VERNON E. HOWES, *all of Fresno State College.* Stresses the understanding of fundamental concepts, and uses a more modern approach wherever feasible. Clearly indicates the use of algebra in more advanced courses. *1960. 286 pages. $4.50.*

## BASIC CONCEPTS OF ELEMENTARY MATHEMATICS

*By* WILLIAM L. SCHAAF, *Brooklyn College.* Captures the spirit of contemporary mathematics and integrates it with those aspects of "classical" mathematics which are pertinent to the elementary school. *1960. 386 pages. $5.50.*

## ELEMENTARY STATISTICS*

*By* PAUL G. HOEL, *University of California.* Designed for a one-semester course in elementary statistics for students with only a limited background in mathematics. Emphasizes the simple classical ideas of sampling rather than the more sophisticated modern notions. *1960. 261 pages. $5.50.*

## THE ANALYSIS OF VARIANCE*

*By* HENRY SCHEFFÉ, *University of California, Berkeley.* Part I elucidates the theory of the analysis of variance in fixed-effects models. Part II treats it in random-effects models, mixed models, and randomization models. *1959. 476 pages. $14.00*

## ANALYTIC GEOMETRY, Third Edition

*By* JOHN W. CELL, *University of North Carolina.* Minimizes memory work, and reinforces fundamentals. Large sections of new and revised problem material have been added in this edition. The varied topics and multiple procedures of analytic geometry which present the necessity of choice and of correlation are covered completely. *1960. 330 pages. $4.95.*

## PROBABILITY AND STATISTICS*

*Edited by* U. GRENANDER, *Brown University.* A series of papers by leading authorities in the field of probability and statistics, dedicated to, and honoring Professor Harald Cramér, Dean of the University of Stockholm. *1959. 434 pages. $12.50.*

***One of the Wiley Publications in Statistics, Walter A. Shewhart and S. S. Wilks, Editors.**

***Send for your examination copies***

**University of Toronto Press** **Toronto, Ontario**